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A THEORY OF ANISOTROPIC VISCOELASTIC SANDWICH SHELLS

by John L. Baylor

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UNIVERSITY OF CALIFORNIA
Berkeley, Calif.
for

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INTRODUCTION

The type of sandwich construction which is considered here consists of two thin anisotropic Kirchhoff-Love shells (facings) separated by a three dimensional orthotropic medium (core) in which the in-plane stresses $\tau^{\alpha\beta}$ [+] are zero, see figure 1. Since $\tau^{\alpha\beta} = 0$ in the core, only the transverse shear resultants $\bar{S}^{\alpha}$ and the mean normal stress σ^{33} need be considered when dealing with the core. On an element of a facing (see figure 1) the force ${}_{\underline{n}}\vec{N}^{\alpha}$ and couple ${}_{\underline{n}}\vec{M}^{\alpha}$ per unit of coordinate are evaluated at the surfaces which are common to both a facing and the core (interfaces).

The prefix $\underline{n}$ stands for $\underline{0}$ or $\underline{1}$ according as the quantity is associated with the upper or lower facing, respectively.

To avoid considering continuity of displacements at the interfaces, the interface displacements (${}_{\underline{n}}\vec{V}$) are utilized. In the formulation of the theory, the sums and differences of the interface displacements ($\bar{w}_r$ and w_r) are introduced.

The dimensionless surface coordinates θ^{α} are assumed to be lines of curvature. Hence, the metric tensors ($a_{\alpha\beta}$ and ${}_{\underline{n}}a_{\alpha\beta}$) and the coefficients of the second fundamental forms ($b_{\alpha\beta}$ and ${}_{\underline{n}}b_{\alpha\beta}$) are diagonal matrices. Also, the coefficients of the second fundamental forms are associated with the curvatures of the surface under consideration.

The differential equations governing the small deflections of the above described sandwich shell are derived from the Hellinger-Reissner

[+]Usual tensor notation prevails, see reference 1.

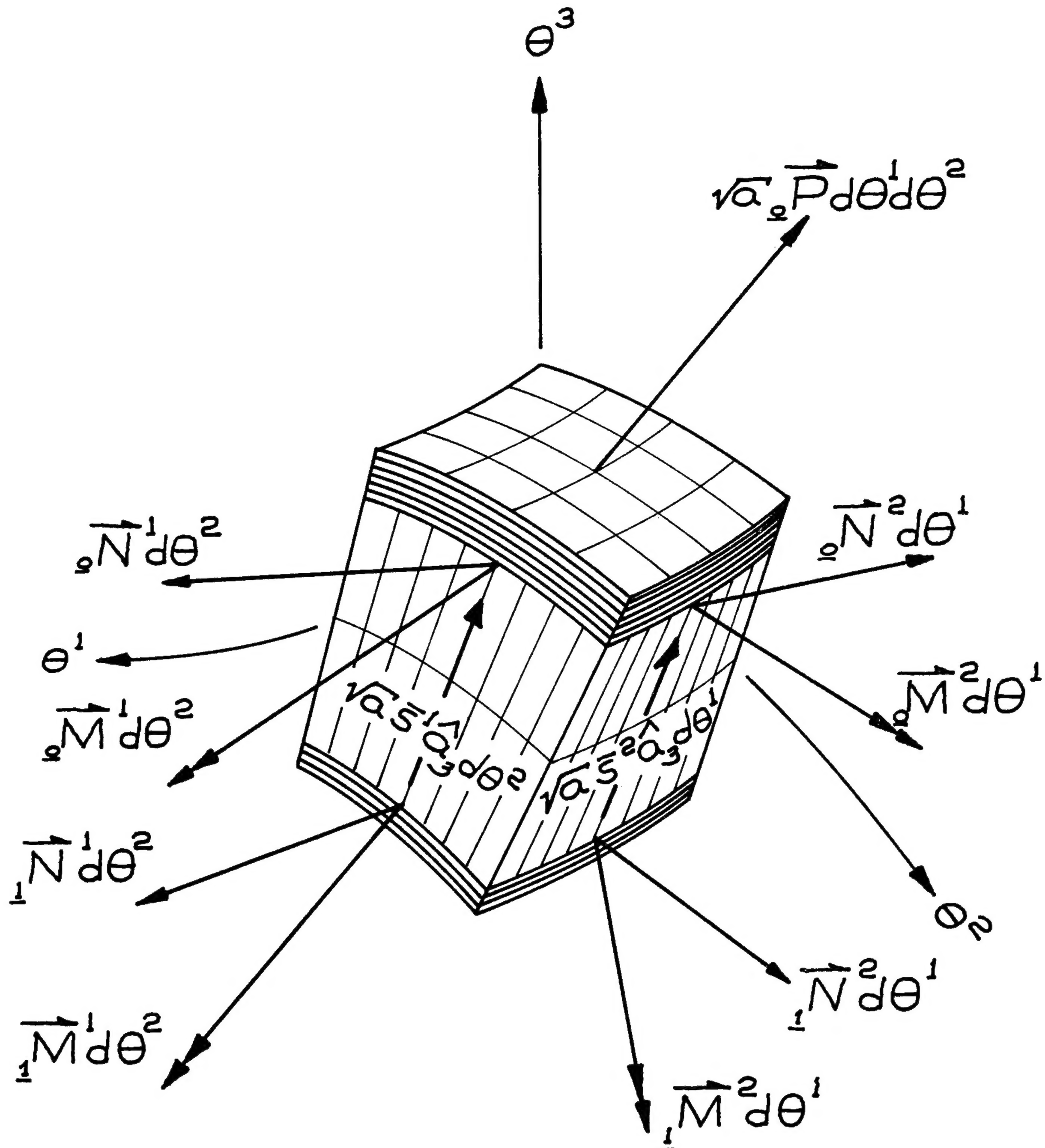

FIG. 1, COMPOSITE SHELL ELEMENT

variational theorem [2][++]. The equilibrium equations for the composite shell are similar to those obtained in [3] and the boundary conditions for the individual facings are comparable to those obtained for a homogeneous shell in [4]. The stress resultant-displacement relations for the composite shell obtained here have not been presented before. Representative equations for a sandwich shell with a viscoelastic core are displayed.

The equations presented here are applicable to plates as well as shells, however, they will not be specialized in their general form since a complete theory of sandwich plates has been given by G. A. Wempner and J. L. Baylor [5].

Many authors have used variational principles in the derivation of sandwich shell theories. E. Reissner [6] and C. T. Wang [7] used the principle of minimum complementary energy to derive the stress resultant-displacement relations for a composite shell. Both Reissner and Wang regarded the facings as membranes. Equations which include the bending stiffness of the individual facings have been derived by E. I. Grigolyuk [8] and R. E. Fulton [9] from the principle of stationary potential energy. A non-variational derivation of sandwich shell theory is given by Wempner and Baylor [3].

Presented here is a theory, developed from the Hellinger-Reissner variational theorem, which includes bending resistance and dissimilarities of the facings. The resulting equations are applied to examples illustrating the effects of anisotropic facings and a viscoelastic core on sandwich shell behavior.

[++]Numbers in brackets refer to the bibliography at the end of the report.

A THEORY OF ANISOTROPIC VISCOELASTIC SANDWICH SHELLS

1. Stress Distribution thru the Core

In what follows it is assumed that the components of the displacement vector and their derivatives are infinitesimals of the first order and the squares and products of these infinitesimals are neglected when compared with their first powers.

The core is weak in the sense that it only resists transverse shear and transverse normal stresses, i.e. $\tau^{\alpha\beta} = 0.$ Upon setting $\tau^{\alpha\beta} = 0$ in the equilibrium equations, the core stresses become statically determinate. Integration of the equilibrium equations gives [3]

$$\sqrt{g}\,\tau^{3\alpha} = \frac{\left[1-(\lambda b^{(\alpha)}_{(\alpha)})^2\right]\sqrt{a}\,\bar{s}^{\alpha}}{2\lambda L\left[1-\lambda\theta^3 b^{(\alpha)}_{(\alpha)}\right]^2} \tag{1}$$

and

$$\sqrt{g}\,\tau^{33} = \frac{L\sqrt{a}}{2\lambda}\sigma^{33} +$$

$$+\left[\frac{\sqrt{a}\,\bar{s}^{\alpha}}{2\lambda L}\left(\frac{\lambda b^{(\alpha)}_{(\alpha)}-\theta^3}{1-\lambda\theta^3 b^{(\alpha)}_{(\alpha)}}\right)\right]_{,\alpha} \tag{2}$$

where σ^{33} and $\bar{s}^{\alpha}/L$ are proportional to physical stress and physical stress resultants, respectively.

On the edge of the core the shear stress distribution is a priori statically determined in terms of the shear resultant. From equilibrium

of a boundary element, the shear resultant on the edge of the core is

$$\mathcal{S} = u_\alpha \bar{s}^\alpha$$

This shear resultant must be assigned on the edge of the core.

2. Core Stress-Strain Relations

The relative displacement of two particles on the normal, one at each interface, is

$$_2\vec{V} - {}_1\vec{V} = \int_{-1}^{+1} V_{r|3} \vec{g}^r d\theta^3.$$

After some manipulation [3] this yields

$$w_3 = \frac{1}{2\lambda L} \int_{-1}^{+1} e_{33} \, d\theta^3 \qquad (3)$$

and

$$w_\alpha = \frac{1}{2L} \Bigg\{ - \left[\frac{V_{3|\alpha}}{1 - \lambda b^{(\alpha)}_{(\alpha)}} \right]_{\theta^3 = +1} - \left[\frac{V_{3|\alpha}}{1 + \lambda b^{(\alpha)}_{(\alpha)}} \right]_{\theta^3 = -1} +$$

$$+ 2 \int_{-1}^{+1} \frac{e_{\alpha 3} \, d\theta^3}{\left[1 - \lambda \theta^3 b^{(\alpha)}_{(\alpha)} \right]^2} + \int_{-1}^{+1} \frac{e_{33,\alpha} \theta^3 d\theta^3}{1 - \lambda \theta^3 b^{(\alpha)}_{(\alpha)}} \Bigg\}. \qquad (4)$$

Presuming the core to be orthotropic with respect to the surface coordinates, the stress and strain components are related as follows;

$$\gamma_{33} = \frac{\lambda^4 L^4}{E} \tau^{33}, \qquad (5)$$

$$\gamma_{3\alpha} = \frac{\lambda^2 L^4 a_{\alpha\beta}}{2 E^{(\beta)}} \left[1 - \lambda \theta^3 b^{(\beta)}_{(\beta)} \right]^2 \tau^{3\beta}. \qquad (6)$$

Because of the displacement assumption

$$\gamma_{33} = e_{33} \tag{7}$$

and

$$\gamma_{3\alpha} = e_{3\alpha} . \tag{8}$$

Substituting, in turn, (7) and (5) into (3) and (7), (8), (5) and (6) into (4), expanding the integrands in λ power series and neglecting λ^2 when compared to one, there results

$$w_3 = \frac{\lambda L}{2E}\left[\sigma^{33} + \frac{2\lambda}{3L^2}\bar{s}^{\alpha}\|_{\alpha}\left(b^{(\alpha)}_{(\alpha)} - h\right) + \right.$$

$$\left. + \frac{2\lambda}{3L^2}\bar{s}^{\alpha} b^{(\alpha)}_{(\alpha),\alpha}\right] \tag{9}$$

and

$$w_\alpha = -\lambda \bar{w}_{3,\alpha} - \lambda^2 w_{3,\alpha} b^{(\alpha)}_{(\alpha)} - \lambda \bar{w}_\alpha b^{(\alpha)}_{(\alpha)} -$$

$$-\lambda^2 w_\alpha \left(b^{(\alpha)}_{(\alpha)}\right)^2 + \frac{a_{\alpha\beta}\bar{s}^{\beta}}{2LE^{(\beta)}} - \frac{\lambda^2}{6LE}\bar{s}^{\beta}\|_{\beta\alpha} +$$

$$+ \frac{\lambda^3 L}{3E}\sigma^{33} h_{,\alpha} + \frac{\lambda^3 L}{6E}\sigma^{33}_{\ \ ,\alpha}\left(b^{(\alpha)}_{(\alpha)} + 2h\right) +$$

$$+ \frac{\lambda^4}{15LE}\left(\bar{s}^{\beta} b^{(\beta)}_{(\beta),\beta}\right)_{,\alpha}\left(b^{(\alpha)}_{(\alpha)} + 2b^{(\beta)}_{(\beta)} + 2h\right) +$$

$$+ \frac{2\lambda^4}{15LE}\bar{s}^{\beta} b^{(\beta)}_{(\beta),\beta}\left(b^{(\beta)}_{(\beta)} + h\right)_{,\alpha} +$$

$$+\frac{\lambda^4}{2LE}\,\bar{s}^{\beta}\|_{\beta}\Big[\tfrac{2}{3}b^{(\beta)}_{(\beta)}h_{,\alpha}-\tfrac{2}{5}\big(b^{(\alpha)}_{(\alpha)}+b^{(\beta)}_{(\beta)}\big)h_{,\alpha}+$$

$$+\tfrac{2}{15}b^{(\beta)}_{(\beta),\alpha}\big(b^{(\alpha)}_{(\alpha)}+2b^{(\beta)}_{(\beta)}+2h\big)-$$

$$-\tfrac{1}{5}\big(8hh_{,\alpha}-k_{,\alpha}\big)\Big]. \qquad (10)$$

Equations (9) and (10) are the core stress-strain relations.

3. Stress Distribution thru a Facing

The force and couple, per unit of coordinate, on an element of a facing are (see figure 1)

$${}_{\underline{n}}\vec{N}^{\alpha}=\sqrt{a}\,{}_{\underline{n}}n^{\alpha\beta}\,{}_{\underline{n}}\vec{a}_{\beta}+\sqrt{a}\,{}_{\underline{n}}q^{\alpha}\hat{a}_3$$

and

$${}_{\underline{n}}\vec{M}^{\alpha}=L\sqrt{a}\,{}_{\underline{n}}\epsilon_{\beta\eta}\,{}_{\underline{n}}m^{\alpha\beta}\,{}_{\underline{n}}\vec{a}^{\eta}$$

where ${}_{\underline{n}}n^{\alpha\beta}$, $L\,{}_{\underline{n}}m^{\alpha\beta}$ and ${}_{\underline{n}}q^{\alpha}$ are proportional to physical stress resultants.

Neglecting terms of order ${}_{\underline{n}}\lambda$, the stress resultants are related to the stresses as follows;

$${}_{\underline{n}}n^{\alpha\beta}=L^4\,{}_{\underline{n}}\lambda\,{}_{\underline{n}}j\int_{\substack{0\\-1}}^{\substack{1\\0}}\tau^{\alpha\beta}d_{\underline{n}}\theta^3, \qquad (11)$$

$${}_{\underline{n}}m^{\alpha\beta}=L^4\,{}_{\underline{n}}\lambda^2\,{}_{\underline{n}}j\int_{\substack{0\\-1}}^{\substack{1\\0}}\tau^{\alpha\beta}_{\underline{n}}\,\theta^3 d_{\underline{n}}\theta^3 \qquad (12)$$

and

$${}_{\underline{n}}q^{\alpha}=L^4\,{}_{\underline{n}}\lambda^2\,{}_{\underline{n}}j\int_{\substack{0\\-1}}^{\substack{1\\0}}\tau^{\alpha 3}d_{\underline{n}}\theta^3. \qquad (13)$$

Notice that ${}_{n}n^{\alpha\beta}$ and ${}_{n}m^{\alpha\beta}$ are symmetric.

Guided by (11) and (12), the stress components are presumed to have the following form

$$\tau^{\alpha\beta} = \left(2 \mp 3\,{}_{n}\theta^{3}\right)\frac{2\,{}_{n}n^{\alpha\beta}}{L^{4}\,{}_{n}\lambda\,{}_{n}j} + \\ + \left(2\,{}_{n}\theta^{3} \mp 1\right)\frac{6\,{}_{n}m^{\alpha\beta}}{L^{4}\,{}_{n}\lambda^{2}\,{}_{n}j} \qquad (14)$$

or

$$\tau^{\alpha\beta} = \left(2 \mp 3\,{}_{n}\theta^{3}\right)\left(\frac{d\bar{n}^{\alpha\beta} \pm 2Ln^{\alpha\beta}}{L^{4}d\,{}_{n}\lambda\,{}_{n}j}\right) + \\ + 3\left(2\,{}_{n}\theta^{3} \mp 1\right)\left(\frac{d\bar{m}^{\alpha\beta} \pm 2Lm^{\alpha\beta}}{L^{4}d\,{}_{n}\lambda^{2}\,{}_{n}j}\right). \qquad (15)$$

The normal stress τ^{33} is assumed to be zero.

4. Strain-Displacement Relations for a Facing

The facings are presumed to be thin Kirchhoff-Love shells, i.e. normals remain straight and normal to the interface surfaces.

If extension of the normal is neglected, the displacement of a particle in a facing is

$$\vec{V} = {}_{n}\vec{V} + {}_{n}\lambda L\,{}_{n}\theta^{3}\left({}_{n}\hat{A}_{3} - \hat{a}_{3}\right). \qquad (16)$$

The deformed and undeformed unit normal vectors are related as follows [3],

$$\underline{n}\hat{A}_3 = \hat{a}_3 + \frac{\underline{n}\omega_{\alpha 3}}{\underline{n}\lambda L^2}\,\underline{n}\vec{a}^{\alpha}. \tag{17}$$

Substituting (17) into (16), one finds

$$\vec{V} = \underline{n}\vec{V} + \underline{n}\theta^3 \frac{\underline{n}\omega_{\alpha 3}}{L}\,\underline{n}\vec{a}^{\alpha}.$$

The covariant components of the displacement vector are

$$V_3 = \frac{\underline{n}\lambda}{\lambda}\,\underline{n}V_3 \tag{18}$$

and

$$V_\beta = \left(\delta^\alpha_\beta - \underline{n}\lambda\,\underline{n}\theta^3_{\underline{n}} b^\alpha_\beta\right)\left(\underline{n}V_\alpha + \underline{n}\theta^3_{\underline{n}}\omega_{\alpha 3}\right) \tag{19}$$

where [3]

$$\underline{n}V_3 = \lambda L\left(\bar{\varpi}_3 \pm \varpi_3\right), \tag{20}$$

$$\underline{n}V_\alpha = L\left(1 \mp \lambda b^{(\alpha)}_{(\alpha)}\right)\left(\bar{\varpi}_\alpha \pm \varpi_\alpha\right) \tag{21}$$

and

$$\underline{n}\omega_{3\alpha} = \underline{n}\lambda\left(\frac{\underline{n}V_{3,\alpha}}{\lambda} + \underline{n}b^\beta_\alpha\,\underline{n}V_\beta\right). \tag{22}$$

Equations (20) and (21) are obtained directly from the definitions of $\bar{\varpi}_r$ and ϖ_r.

Because of the displacement assumption [1]

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(V_{\alpha|\beta} + V_{\beta|\alpha}\right) \tag{23}$$

If one uses (18), (19), (20), (21) and (22) in (23) and neglects terms of order $\underline{n}\lambda$, then

$$\gamma_{\alpha\beta} = {}_{\underline{n}}\gamma_{\alpha\beta} - {}_{\underline{n}}\theta^3\, {}_{\underline{n}}\varkappa_{\alpha\beta} \tag{24}$$

where

$$\begin{aligned} {}_{\underline{n}}\gamma_{\alpha\beta} = \frac{1}{2}\Big[& {}_{\underline{n}}V_{\alpha,\beta} + {}_{\underline{n}}V_{\beta,\alpha} - 2\, {}_{\underline{n}}\Gamma^{\gamma}_{\alpha\beta}\, {}_{\underline{n}}V_{\gamma} - \\ & - 2\frac{{}_{\underline{n}}b_{\alpha\beta}}{\lambda}\, {}_{\underline{n}}V_3 \Big] \end{aligned} \tag{25}$$

and

$${}_{\underline{n}}\varkappa_{\alpha\beta} = \frac{1}{2}\Big[{}_{\underline{n}}\omega_{3\alpha,\beta} + {}_{\underline{n}}\omega_{3\beta,\alpha} - 2\, {}_{\underline{n}}\Gamma^{\gamma}_{\alpha\beta}\, {}_{\underline{n}}\omega_{3\gamma} \Big] \tag{26}$$

Equations (24), (25) and (26) are the same strain-displacement relations obtained in [3]. Also ${}_{\underline{n}}\gamma_{\alpha\beta}$ and ${}_{\underline{n}}\varkappa_{\alpha\beta}$ as defined here agree with $\mathring{\gamma}_{\alpha\beta}$ and $\rho_{\alpha\beta}$ of [10].

The shear strain $\gamma_{3\alpha}$ is zero at the interfaces and will be assumed zero throughout a facing.

5. Hellinger-Reissner Three Dimensional Variational Theorem

The equilibrium equations, stress resultant-displacement relations and boundary conditions for a sandwich shell will be derived from the following variational principle of Hellinger and Reissner [2].

The state of stress and displacement which satisfies the differential equations of equilibrium and the stress displacement relations in the interior of the body, and the conditions of prescribed stress on the part σ_1 and of prescribed displacements on the part σ_2 of the surface of the body, is determined by the variational equation

$$\delta\Big\{ \iiint_{\tau} \left(\tau^{rs}\gamma_{rs} - W \right) d\tau - \iint_{\sigma_1} \tilde{p}^{r} V_r \, d\sigma -$$

$$-\iint_{\sigma_2}(V_r-\tilde{V}_r)P^r d\sigma \Big\} = 0. \qquad (27)$$

6. Contribution to the Hellinger-Reissner Theorem from the Facings

The normal stress τ^{33} and the shear strains $\gamma_{3\alpha}$ have been assumed zero. Having zero shear strains, $\gamma_{3\alpha}$, while there exist non-zero inplane stresses $\tau^{\alpha\beta}$ and non-zero shear stresses $\tau^{3\alpha}$ requires the elastic coefficients ${}_{n}C_{3\alpha\beta\gamma}$ and ${}_{n}C_{3\alpha3\beta}$ to be zero. Thus the only contributions from the facings to the volume integral of the variational theorem are

$$\iint_{{}_{2}S}\left[\int_{0}^{1}\tau^{\alpha\beta}\gamma_{\alpha\beta}\sqrt{{}_{2}g}\,d_{2}\theta^{3}\right]d\theta^{1}d\theta^{2}+$$

$$+\iint_{{}_{1}S}\left[\int_{-1}^{0}\tau^{\alpha\beta}\gamma_{\alpha\beta}\sqrt{{}_{1}g}\,d_{1}\theta^{3}\right]d\theta^{1}d\theta^{2} \qquad (28)$$

and

$$\iint_{{}_{2}S}\left[\int_{0}^{1}{}_{2}C_{\alpha\beta\gamma\eta}\tau^{\alpha\beta}\tau^{\gamma\eta}\sqrt{{}_{2}g}\,d_{2}\theta^{3}\right]d\theta^{1}d\theta^{2}+$$

$$+\iint_{{}_{1}S}\left[\int_{-1}^{0}{}_{1}C_{\alpha\beta\gamma\eta}\tau^{\alpha\beta}\tau^{\gamma\eta}\sqrt{{}_{1}g}\,d_{1}\theta^{3}\right]d\theta^{1}d\theta^{2}. \qquad (29)$$

Substituting (15) and (24) into the integrals thru a facing thickness and neglecting terms of order ${}_{n}\lambda$, one finds

$$\int_{\substack{0\\-1}}^{\substack{1\\0}}\tau^{\alpha\beta}\gamma_{\alpha\beta}d_{n}\theta^{3}=\frac{d\bar{n}^{\alpha\beta}\pm 2Ln^{\alpha\beta}}{2L^{3}d_{n}d_{n}j}\,{}_{n}\gamma_{\alpha\beta}-$$

$$
-\frac{d\bar{m}^{\alpha\beta} \pm 2Lm^{\alpha\beta}}{2L^2 d_{n} d^2_{n} j}\, {}_{n}\kappa_{\alpha\beta} \tag{30}
$$

and

$$
\int_{-1/0}^{1/0} \tau^{\alpha\beta}\tau^{\gamma\eta}\, d_{n}\theta^3 =
$$

$$
= \frac{1}{L^6 d^2_{n} d^2_{n} j^2}\left[d\bar{n}^{\alpha\beta} \pm 2Ln^{\alpha\beta}\right]\left[d\bar{n}^{\gamma\eta} \pm 2Ln^{\gamma\eta}\right] +
$$

$$
+ \frac{3}{2L^5 d^2_{n} d^3_{n} j^2}\left[d\bar{n}^{\alpha\beta} \pm 2Ln^{\alpha\beta}\right]\left[d\bar{m}^{\gamma\eta} \pm 2Lm^{\gamma\eta}\right] +
$$

$$
+ \frac{3}{2L^5 d^2_{n} d^3_{n} j^2}\left[d\bar{n}^{\gamma\eta} \pm 2Ln^{\gamma\eta}\right]\left[d\bar{m}^{\alpha\beta} \pm 2Lm^{\alpha\beta}\right] +
$$

$$
+ \frac{3}{L^4 d^2_{n} d^4_{n} j^2}\left[d\bar{m}^{\alpha\beta} \pm 2Lm^{\alpha\beta}\right]\left[d\bar{m}^{\gamma\eta} \pm 2Lm^{\gamma\eta}\right] \tag{31}
$$

The surface of a facing consists of three parts; the interface surface, the exterior face and the edge. Over the exterior face and the interface $\sigma_2 = 0$, i.e., stresses are prescribed. The integrals

$$
\iint_{{}_{n}S} \tilde{P}^r V_r \, d_{n}S ,
$$

over the interfaces, from the facings are the negatives of the corresponding integrals from the core, consequently they sum to zero.

The load on the exterior face of a facing is

$$
{}_{n}\vec{P} = \left[{}_{n}p^{\alpha}_{n}\vec{a}_{\alpha} + {}_{n}p^3 \hat{a}_3\right]\frac{1}{L^2_{n} j} \tag{32}
$$

where ${}_{\Omega}\mathcal{P}^{\alpha}/L^2$ and ${}_{\Omega}\mathcal{P}^{3}/L^2$ are proportional to physical force per unit undeformed area.

Using (18), (19), (20), (21) and (32), neglecting surface load times rotation terms and neglecting terms of order ${}_{\Omega}\lambda$ and λ^2 one obtains

$$\iint_{{}_{\Omega}\bar{S}} \tilde{P}^{r} V_r \, d_{\Omega}\bar{S} = \frac{1}{2}\iint_{S}\Big[\big(1 \mp \lambda b^{(\alpha)}_{(\alpha)}\big)\big(\bar{\mathcal{P}}^{\alpha} \pm \mathcal{P}^{\alpha}\big)\cdot$$

$$\cdot\big(\bar{w}_{\alpha} \pm w_{\alpha}\big) + \big(\bar{\mathcal{P}}^{3} \pm \mathcal{P}^{3}\big)\big(\bar{w}_{3} \pm w_{3}\big)\Big] dS. \qquad (33)$$

From (13), (14), (18) and (19), after neglecting terms of order ${}_{\Omega}\lambda$ and λ^2 it is found that

$$\iint_{{}_{\Omega}\Omega_1} \tilde{P}^{r} V_r \, d_{\Omega}\Omega + \iint_{{}_{\Omega}\Omega_2} \big(V_r - \tilde{V}_r\big) P^{r} d_{\Omega}\Omega =$$

$$= \int_{{}_{\Omega}C_1}\Big[\frac{1}{L}\,{}_{\Omega}\tilde{n}^{\alpha\beta}\,{}_{\Omega}V_{\beta} - \frac{2}{d}\,{}_{\Omega}\tilde{m}^{\alpha\beta}\,{}_{\Omega}V_{3,\beta} + \frac{2}{d}\,{}_{\Omega}\tilde{q}^{\alpha}\,{}_{\Omega}V_3 -$$

$$- \frac{1}{L} b^{(\beta)}_{(\beta)}\big(1 \pm \lambda b^{(\beta)}_{(\beta)}\big)\,{}_{\Omega}\tilde{m}^{\alpha\beta}\,{}_{\Omega}V_{\beta}\Big]\frac{{}_{\Omega}u_{\alpha}}{{}_{\Omega}j}\, d_{\Omega}s +$$

$$+ \int_{{}_{\Omega}C_2}\Big[\frac{1}{L}\,{}_{\Omega}n^{\alpha\beta}\big({}_{\Omega}V_{\beta} - {}_{\Omega}\tilde{V}_{\beta}\big) - \frac{2}{d}\,{}_{\Omega}m^{\alpha\beta}\big({}_{\Omega}V_{3,\beta} - {}_{\Omega}\tilde{V}_{3,\beta}\big) -$$

$$- \frac{1}{L} b^{(\beta)}_{(\beta)}\big(1 \pm \lambda b^{(\beta)}_{(\beta)}\big)\,{}_{\Omega}m^{\alpha\beta}\big({}_{\Omega}V_{\beta} - {}_{\Omega}\tilde{V}_{\beta}\big) +$$

$$+ \frac{2}{d}\,{}_{\Omega}q^{\alpha}\big({}_{\Omega}V_3 - {}_{\Omega}\tilde{V}_3\big)\Big]\frac{{}_{\Omega}u_{\alpha}}{{}_{\Omega}j}\, d_{\Omega}s. \qquad (34)$$

This is the contribution to the surface integrals from the edge of a facing.

7. Contribution to the Hellinger-Reissner Theorem from the Core

For the core the equilibrium equations have been identically satisfied, the stress-strain relations have been determined, the boundary condition has been obtained and over the interface surfaces σ_2 has been presumed zero. Thus the only contribution to the variational theorem from the core is

$$\iint_{{}_{n}S} P^{r} \delta V_{r} \, d_{n}S .$$

The outward unit normals to the core interface surfaces are

$$ {}_{n}\hat{n} = \pm \lambda L \vec{g}^{3}. \qquad (35)$$

Equations (1), (2), (20), (21) and (35) give

$$\iint_{{}_{n}S} P^{r} \delta V_{r} \, d_{n}S = \iint_{S} \Big[\Big(\pm \frac{L^2}{2} \sigma^{33} - \frac{1}{2} \bar{S}^{\alpha} \|_{\alpha} \Big) \cdot$$

$$\cdot \left(\delta \bar{w}_3 \pm \delta w_3 \right) + \frac{\bar{S}^{\alpha}}{2\lambda} \left(\pm 1 + \lambda b^{(\alpha)}_{(\alpha)} \right) \left(\delta \bar{w}_{\alpha} \pm \right.$$

$$\left. \pm \delta w_{\alpha} \right) \Big] dS. \qquad (36)$$

8. Hellinger-Reissner Variational Theorem for a Sandwich Shell

Upon substituting (28), (29), (30), (31), (33), (34) and (36) into the variational theorem (27) and using Green's theorem [1], one obtains the following variational equation appropriate for a sandwich shell.

$$
\begin{aligned}
&\iint_S \Big\{ \delta\bar{n}^{\alpha\beta}\Big[\Big(\bar{w}_\alpha - \lambda b^{(\alpha)}_{(\alpha)} w_\alpha\Big)_{\|\beta} - \lambda \bar{\Gamma}^{\gamma}_{\alpha\beta}\Big(\bar{w}_\gamma - \\
&-\lambda b^{(\gamma)}_{(\gamma)} w_\gamma\Big) - \lambda^2 \Gamma^{\gamma}_{\alpha\beta}\Big(w_\gamma - \lambda b^{(\gamma)}_{(\gamma)} \bar{w}_\gamma\Big) - b_{\alpha\beta}\Big(\bar{w}_3 - \\
&-\lambda b^{(\alpha)}_{(\alpha)} w_3\Big) - \bar{C}_{\alpha\beta\gamma\eta}\Big(\frac{\bar{\alpha}}{L}\bar{n}^{\gamma\eta} + \frac{2\alpha}{d} n^{\gamma\eta} - \frac{3\beta}{2L}\bar{m}^{\gamma\eta} - \\
&-\frac{3\bar{\beta}}{d} m^{\gamma\eta}\Big) - C_{\alpha\beta\gamma\eta}\Big(\frac{\alpha}{L}\bar{n}^{\gamma\eta} + \frac{2\bar{\alpha}}{d} n^{\gamma\eta} - \frac{3\bar{\beta}}{2L}\bar{m}^{\gamma\eta} - \\
&-\frac{3\beta}{d} m^{\gamma\eta}\Big)\Big] + \delta n^{\alpha\beta}\Big[\frac{1}{\lambda}\Big(w_\alpha - \lambda b^{(\alpha)}_{(\alpha)} \bar{w}_\alpha\Big)_{\|\beta} - \\
&-\bar{\Gamma}^{\gamma}_{\alpha\beta}\Big(w_\gamma - \lambda b^{(\gamma)}_{(\gamma)} \bar{w}_\gamma\Big) - \lambda \Gamma^{\gamma}_{\alpha\beta}\Big(\bar{w}_\gamma - \lambda b^{(\gamma)}_{(\gamma)} w_\gamma\Big) - \\
&-\frac{1}{\lambda} b_{\alpha\beta}\Big(w_3 - \lambda b^{(\alpha)}_{(\alpha)} \bar{w}_3\Big) - \frac{1}{\lambda}\bar{C}_{\alpha\beta\gamma\eta}\Big(\frac{\bar{\alpha}}{L}\bar{n}^{\gamma\eta} + \\
&+\frac{2\bar{\alpha}}{d} n^{\gamma\eta} - \frac{3\bar{\beta}}{2L}\bar{m}^{\gamma\eta} - \frac{3\beta}{d} m^{\gamma\eta}\Big) - \frac{1}{\lambda} C_{\alpha\beta\gamma\eta}\Big(\frac{\bar{\alpha}}{L}\bar{n}^{\gamma\eta} + \\
&+\frac{2\alpha}{d} n^{\gamma\eta} - \frac{3\beta}{2L}\bar{m}^{\gamma\eta} - \frac{3\bar{\beta}}{d} m^{\gamma\eta}\Big)\Big] + \delta\bar{m}^{\alpha\beta}\Big[-\Big(\bar{w}_{3,\alpha} + \\
&+\bar{w}_\alpha b^{(\alpha)}_{(\alpha)}\Big)_{\|\beta} + \lambda \bar{\Gamma}^{\gamma}_{\alpha\beta}\Big(\bar{w}_{3,\gamma} + \bar{w}_\gamma b^{(\gamma)}_{(\gamma)}\Big) + \\
&+\lambda^2 \Gamma^{\gamma}_{\alpha\beta}\Big(w_{3,\gamma} + w_\gamma b^{(\gamma)}_{(\gamma)}\Big) + \bar{C}_{\alpha\beta\gamma\eta}\Big(\frac{3\beta}{2L}\bar{n}^{\gamma\eta} +
\end{aligned}
$$

$$+\frac{3\bar{\beta}}{d}n^{\gamma\eta}-\frac{3\gamma}{L}\bar{m}^{\gamma\eta}-\frac{6\bar{\gamma}}{d}m^{\gamma\eta}\Big)+C_{\alpha\beta\gamma\eta}\Big(\frac{3\bar{\beta}}{2L}\bar{n}^{\gamma\eta}+$$

$$+\frac{3\beta}{d}n^{\gamma\eta}-\frac{3\bar{\gamma}}{L}\bar{m}^{\gamma\eta}-\frac{6\gamma}{d}m^{\gamma\eta}\Big)\Big]+\delta m^{\alpha\beta}\Big[-\frac{1}{\lambda}\Big(w_{3,\alpha}+$$

$$+w_{\alpha}b^{(\alpha)}_{(\alpha)}\Big)\Big\|_{\beta}+\bar{\Upsilon}^{\gamma}_{\alpha\beta}\Big(w_{3,\gamma}+w_{\gamma}b^{(\gamma)}_{(\gamma)}\Big)+\lambda\,\Upsilon^{\gamma}_{\alpha\beta}\Big(\bar{w}_{3,\gamma}+$$

$$+\bar{w}_{\gamma}b^{(\gamma)}_{(\gamma)}\Big)+\frac{1}{\lambda}\bar{C}_{\alpha\beta\gamma\eta}\Big(\frac{3\bar{\beta}}{2L}\bar{n}^{\gamma\eta}+\frac{3\beta}{d}n^{\gamma\eta}-\frac{3\gamma}{L}\bar{m}^{\gamma\eta}-$$

$$-\frac{6\bar{\gamma}}{d}m^{\gamma\eta}\Big)+\frac{1}{\lambda}C_{\alpha\beta\gamma\eta}\Big(\frac{3\beta}{2L}\bar{n}^{\gamma\eta}+\frac{3\bar{\beta}}{d}n^{\gamma\eta}-\frac{3\bar{\gamma}}{L}\bar{m}^{\gamma\eta}-$$

$$-\frac{6\gamma}{d}m^{\gamma\eta}\Big)\Big]+\delta\bar{w}_{\alpha}\Big[-\bar{n}^{\alpha\beta}\|_{\beta}+b^{(\alpha)}_{(\alpha)}n^{\alpha\beta}\|_{\beta}-\lambda\bar{\Upsilon}^{\alpha}_{\gamma\beta}\Big(\bar{n}^{\gamma\beta}-$$

$$-b^{(\alpha)}_{(\alpha)}n^{\gamma\beta}\Big)+\lambda\Upsilon^{\alpha}_{\gamma\beta}\Big(\lambda^{2}b^{(\alpha)}_{(\alpha)}\bar{n}^{\gamma\beta}-n^{\gamma\beta}\Big)+\bar{s}^{\alpha}b^{(\alpha)}_{(\alpha)}+$$

$$+b^{(\alpha)}_{(\alpha)}\bar{m}^{\alpha\beta}\|_{\beta}+\lambda b^{(\alpha)}_{(\alpha)}\bar{\Upsilon}^{\alpha}_{\gamma\beta}\bar{m}^{\gamma\beta}+\lambda b^{(\alpha)}_{(\alpha)}\Upsilon^{\alpha}_{\gamma\beta}\,m^{\gamma\beta}-$$

$$-\bar{p}^{\alpha}+\lambda b^{(\alpha)}_{(\alpha)}p^{\alpha}\Big]+\delta w_{\alpha}\Big[-\frac{1}{\lambda}n^{\alpha\beta}\|_{\beta}+\lambda b^{(\alpha)}_{(\alpha)}\bar{n}^{\alpha\beta}\|_{\beta}-$$

$$-\bar{\Upsilon}^{\alpha}_{\gamma\beta}\Big(n^{\gamma\beta}-\lambda^{2}b^{(\alpha)}_{(\alpha)}\bar{n}^{\gamma\beta}\Big)+\lambda^{2}\Upsilon^{\alpha}_{\gamma\beta}\Big(b^{(\alpha)}_{(\alpha)}n^{\gamma\beta}-\bar{n}^{\gamma\beta}\Big)+$$

$$+\frac{1}{\lambda}\bar{s}^{\alpha}+\frac{1}{\lambda}b^{(\alpha)}_{(\alpha)}m^{\alpha\beta}\|_{\beta}+b^{(\alpha)}_{(\alpha)}\bar{\Upsilon}^{\alpha}_{\gamma\beta}m^{\gamma\beta}+$$

$$+\lambda^2 b^{(\alpha)}_{(\alpha)}\Upsilon^{\alpha}_{\gamma\beta}\overline{m}^{\gamma\beta}-p^{\alpha}+\lambda b^{(\alpha)}_{(\alpha)}\overline{p}^{\alpha}\Big]+\delta\overline{w}_3\Big[-b_{\alpha\beta}\overline{n}^{\alpha\beta}+$$

$$+b^{(\alpha)}_{(\alpha)}b_{\alpha\beta}n^{\alpha\beta}-\overline{s}^{\alpha}\|_{\alpha}-\overline{p}^{3}-\overline{m}^{\alpha\beta}\|_{\alpha\beta}-\lambda\Big(\overline{\Upsilon}^{\gamma}_{\alpha\beta}\overline{m}^{\alpha\beta}+$$

$$+\Upsilon^{\gamma}_{\alpha\beta}m^{\alpha\beta}\Big)\Big\|_{\gamma}\Big]+\delta w_3\Big[-\frac{1}{\lambda}b_{\alpha\beta}n^{\alpha\beta}+\lambda b^{(\alpha)}_{(\alpha)}b_{\alpha\beta}\overline{n}^{\alpha\beta}-$$

$$-p^{3}-\frac{1}{\lambda}m^{\alpha\beta}\|_{\alpha\beta}+L^2\sigma^{33}-\Big(\overline{\Upsilon}^{\gamma}_{\alpha\beta}m^{\alpha\beta}+$$

$$+\lambda^2\Upsilon^{\gamma}_{\alpha\beta}m^{\alpha\beta}\Big)\Big\|_{\gamma}\Big]\Big\}ds+\sum_{n=0}^{1}\int_{{}_{n}C_1}\Big\{\frac{1}{L}\Big({}_{n}n^{\alpha\beta}-$$

$$-{}_{n}\tilde{n}^{\alpha\beta}\Big)\delta_{n}V_{\beta}-\frac{2}{d}\Big({}_{n}m^{\alpha\beta}-{}_{n}\tilde{m}^{\alpha\beta}\Big)\delta_{n}V_{3,\beta}+$$

$$+\frac{2}{d}\Big({}_{n}q^{\alpha}-{}_{n}\tilde{q}^{\alpha}\Big)\delta_{n}V_{3}-\frac{1}{L}b^{(\beta)}_{(\beta)}\Big(1\pm\lambda b^{(\beta)}_{(\beta)}\Big)\Big({}_{n}m^{\alpha\beta}-$$

$$-{}_{n}\tilde{m}^{\alpha\beta}\Big)\delta_{n}V_{\beta}\Big\}\frac{{}_{n}u_{\alpha}}{{}_{n}j}d_{n}s-\sum_{n=0}^{1}\int_{{}_{n}C_2}\Big\{\frac{1}{L}\Big({}_{n}V_{\beta}-$$

$$-{}_{n}\tilde{V}_{\beta}\Big)\delta_{n}n^{\alpha\beta}-\frac{2}{d}\Big({}_{n}V_{3,\beta}-{}_{n}\tilde{V}_{3,\beta}\Big)\delta_{n}m^{\alpha\beta}+$$

$$+\frac{2}{d}\Big({}_{n}V_{3}-{}_{n}\tilde{V}_{3}\Big)\delta_{n}q^{\alpha}-\frac{1}{L}b^{(\beta)}_{(\beta)}\Big(1\pm\lambda b^{(\beta)}_{(\beta)}\Big)\Big({}_{n}V_{\beta}-$$

$$-{}_{n}\tilde{V}_{\beta}\Big)\delta_{n}m^{\alpha\beta}\Big\}\frac{{}_{n}u_{\alpha}}{{}_{n}j}d_{n}s=0. \tag{37}$$

In the boundary integrals of (37) the following moment equilibrium equation was used;

$$
{}_{\underline{n}}q^{\alpha} = {}_{\underline{n}}m^{\alpha\beta}\|_{\beta} + \lambda\,{}_{\underline{n}}\Upsilon^{\alpha}_{\beta\gamma}\,{}_{\underline{n}}m^{\beta\gamma}.
$$

This equation is derived in [3].

Equation (37) is the required variational equation for a composite sandwich shell.

9. Equilibrium Equations

The Euler equations resulting from operating on (37) and corresponding to $\delta\bar{w}_3$ and δw_3 are

$$
\begin{aligned}
&\bar{p}^{3} + \bar{s}^{\alpha}\|_{\alpha} + b_{\alpha\beta}\bar{n}^{\alpha\beta} + \bar{m}^{\alpha\beta}\|_{\alpha\beta} - b^{(\alpha)}_{(\alpha)}b_{\alpha\beta}n^{\alpha\beta} + \\
&+ \lambda\left(\bar{\Upsilon}^{\gamma}_{\alpha\beta}\bar{m}^{\alpha\beta} + \Upsilon^{\gamma}_{\alpha\beta}m^{\alpha\beta}\right)\|_{\gamma} = 0
\end{aligned} \tag{38}
$$

and

$$
\begin{aligned}
&-\lambda L^{2}\sigma^{33} + \lambda p^{3} + b_{\alpha\beta}n^{\alpha\beta} + m^{\alpha\beta}\|_{\alpha\beta} - \\
&-\lambda^{2}b^{(\alpha)}_{(\alpha)}b_{\alpha\beta}\bar{n}^{\alpha\beta} + \lambda\left(\bar{\Upsilon}^{\gamma}_{\alpha\beta}m^{\alpha\beta} + \right. \\
&\left. +\lambda^{2}\Upsilon^{\gamma}_{\alpha\beta}\bar{m}^{\alpha\beta}\right)\|_{\gamma} = 0.
\end{aligned} \tag{39}
$$

Forming linear combinations of the Euler equations associated with $\delta\bar{w}_{\alpha}$ and δw_{α}, one finds

$$
\begin{aligned}
&\bar{p}^{\alpha} + \bar{n}^{\alpha\beta}\|_{\beta} - 2\bar{s}^{\alpha}b^{(\alpha)}_{(\alpha)} + \lambda\Upsilon^{\alpha}_{\gamma\beta}\bar{n}^{\gamma\beta} + \\
&+\lambda\Upsilon^{\alpha}_{\gamma\beta}n^{\gamma\beta} - b^{(\alpha)}_{(\alpha)}\bar{m}^{\alpha\beta}\|_{\beta} - \left(b^{(\alpha)}_{(\alpha)}\right)^{2}m^{\alpha\beta}\|_{\beta} - \\
&-\lambda\bar{m}^{\gamma\beta}\left[\bar{\Upsilon}^{\alpha}_{\gamma\beta}b^{(\alpha)}_{(\alpha)} + \lambda^{2}\Upsilon^{\alpha}_{\gamma\beta}\left(b^{(\alpha)}_{(\alpha)}\right)^{2}\right] - \\
&-\lambda m^{\gamma\beta}\left[\Upsilon^{\alpha}_{\gamma\beta}b^{(\alpha)}_{(\alpha)} + \bar{\Upsilon}^{\alpha}_{\gamma\beta}\left(b^{(\alpha)}_{(\alpha)}\right)^{2}\right] = 0
\end{aligned} \tag{40}
$$

and

$$\lambda p^{\alpha} - \bar{s}^{\alpha} + n^{\alpha\beta}\|_{\beta} + \lambda \bar{\Upsilon}^{\alpha}_{\gamma\beta} n^{\gamma\beta} + \lambda^3 \Upsilon^{\alpha}_{\gamma\beta} \bar{n}^{\gamma\beta} -$$
$$- b^{(\alpha)}_{(\alpha)} m^{\alpha\beta}\|_{\beta} - \lambda^2 \left(b^{(\alpha)}_{(\alpha)}\right)^2 \bar{m}^{\alpha\beta}\|_{\beta} -$$
$$-\lambda m^{\gamma\beta}\left[\bar{\Upsilon}^{\alpha}_{\gamma\beta} b^{(\alpha)}_{(\alpha)} + \lambda^2 \Upsilon^{\alpha}_{\gamma\beta}\left(b^{(\alpha)}_{(\alpha)}\right)^2\right] -$$
$$-\lambda^3 \bar{m}^{\gamma\beta}\left[\Upsilon^{\alpha}_{\gamma\beta} b^{(\alpha)}_{(\alpha)} + \bar{\Upsilon}^{\alpha}_{\gamma\beta}\left(b^{(\alpha)}_{(\alpha)}\right)^2\right] = 0. \qquad (41)$$

Equations (38), (39), (40) and (41) are the equilibrium equations for a composite sandwich shell. If the equilibrium equations of [3] are specialized to small rotations and if λ^2 is neglected when compared to one, the resulting equations are the same as (38), (39), (40) and (41).

Equations (38) and (40) are identified with the equilibrium of a gross element of the composite shell.

10. Stress Resultant-Displacement Relations

Combining the Euler equations corresponding to $\delta\bar{n}^{\alpha\beta}$, $\delta n^{\alpha\beta}$, $\delta\bar{m}^{\alpha\beta}$ and $\delta m^{\alpha\beta}$ in a suitable way, it can be verified that

$$\bar{n}^{\xi\mu} = {}_2j_1 j_2 \lambda_1 \lambda L^9 \bar{B}^{\xi\mu\alpha\beta}\left[\bar{\alpha}\left(\bar{w}_{\alpha} - \lambda b^{(\alpha)}_{(\alpha)} w_{\alpha}\right)\|_{\beta} -\right.$$
$$-\alpha\left(w_{\alpha} - \lambda b^{(\alpha)}_{(\alpha)} \bar{w}_{\alpha}\right)\|_{\beta} - \bar{\alpha} b_{\alpha\beta}\left(\bar{w}_3 - \lambda b^{(\alpha)}_{(\alpha)} w_3\right) +$$
$$+\alpha b_{\alpha\beta}\left(w_3 - \lambda b^{(\alpha)}_{(\alpha)} \bar{w}_3\right) +$$
$$+\lambda\left(-\bar{\alpha}\bar{\Upsilon}^{\gamma}_{\alpha\beta} + \lambda\alpha\Upsilon^{\gamma}_{\alpha\beta}\right)\left(\bar{w}_{\gamma} - \lambda b^{(\gamma)}_{(\gamma)} w_{\gamma}\right) +$$
$$\left.+\lambda\left(\alpha\bar{\Upsilon}^{\gamma}_{\alpha\beta} - \lambda\bar{\alpha}\Upsilon^{\gamma}_{\alpha\beta}\right)\left(w_{\gamma} - \lambda b^{(\gamma)}_{(\gamma)} \bar{w}_{\gamma}\right)\right] +$$

$$
\begin{aligned}
&+{}_{\underline{0}}j_{\underline{1}}j_{\underline{0}}\lambda_{\underline{1}}\lambda L^{9}B^{\xi\mu\alpha\beta}\Big[-\alpha\big(\overline{\varpi}_{\alpha}-\lambda b^{(\alpha)}_{(\alpha)}\varpi_{\alpha}\big)_{\|\beta}+\\
&+\overline{\alpha}\big(\varpi_{\alpha}-\lambda b^{(\alpha)}_{(\alpha)}\overline{\varpi}_{\alpha}\big)_{\|\beta}+\alpha b_{\alpha\beta}\big(\overline{\varpi}_{3}-\lambda b^{(\alpha)}_{(\alpha)}\varpi_{3}\big)-\\
&-\overline{\alpha}b_{\alpha\beta}\big(\varpi_{3}-\lambda b^{(\alpha)}_{(\alpha)}\overline{\varpi}_{3}\big)+\\
&+\lambda\big(\alpha\overline{\Upsilon}^{\gamma}_{\alpha\beta}-\lambda\overline{\alpha}\Upsilon^{\gamma}_{\alpha\beta}\big)\big(\overline{\varpi}_{\gamma}-\lambda b^{(\gamma)}_{(\gamma)}\varpi_{\gamma}\big)+\\
&+\lambda\big(-\overline{\alpha}\overline{\Upsilon}^{\gamma}_{\alpha\beta}+\lambda\alpha\Upsilon^{\gamma}_{\alpha\beta}\big)\big(\varpi_{\gamma}-\lambda b^{(\gamma)}_{(\gamma)}\overline{\varpi}_{\gamma}\big)\Big]+\\
&+\frac{1}{2}{}_{\underline{0}}j_{\underline{1}}j_{\underline{0}}\lambda^{2}_{\underline{1}}\lambda^{2}L^{9}\overline{B}^{\xi\mu\alpha\beta}\Big[\beta\big(\overline{\varpi}_{3,\alpha}+\\
&+\overline{\varpi}_{\alpha}b^{(\alpha)}_{(\alpha)}\big)_{\|\beta}-\overline{\beta}\big(\varpi_{3,\alpha}+\varpi_{\alpha}b^{(\alpha)}_{(\alpha)}\big)_{\|\beta}+\\
&+\lambda\big(-\beta\overline{\Upsilon}^{\gamma}_{\alpha\beta}+\lambda\overline{\beta}\Upsilon^{\gamma}_{\alpha\beta}\big)\big(\overline{\varpi}_{3,\gamma}+\overline{\varpi}_{\gamma}b^{(\gamma)}_{(\gamma)}\big)+\\
&+\lambda\big(\overline{\beta}\overline{\Upsilon}^{\gamma}_{\alpha\beta}-\lambda\beta\Upsilon^{\gamma}_{\alpha\beta}\big)\big(\varpi_{3,\gamma}+\varpi_{\gamma}b^{(\gamma)}_{(\gamma)}\big)\Big]+\\
&+\frac{1}{2}{}_{\underline{0}}j_{\underline{1}}j_{\underline{0}}\lambda^{2}_{\underline{1}}\lambda^{2}L^{9}B^{\xi\mu\alpha\beta}\Big[-\overline{\beta}\big(\overline{\varpi}_{3,\alpha}+\\
&+\overline{\varpi}_{\alpha}b^{(\alpha)}_{(\alpha)}\big)_{\|\beta}+\beta\big(\varpi_{3,\alpha}+\varpi_{\alpha}b^{(\alpha)}_{(\alpha)}\big)_{\|\beta}+\\
&+\lambda\big(\overline{\beta}\overline{\Upsilon}^{\gamma}_{\alpha\beta}-\lambda\beta\Upsilon^{\gamma}_{\alpha\beta}\big)\big(\overline{\varpi}_{3,\gamma}+\overline{\varpi}_{\gamma}b^{(\gamma)}_{(\gamma)}\big)+\\
&+\lambda\big(-\beta\overline{\Upsilon}^{\gamma}_{\alpha\beta}+\lambda\overline{\beta}\Upsilon^{\gamma}_{\alpha\beta}\big)\big(\varpi_{3,\gamma}+\varpi_{\gamma}b^{(\gamma)}_{(\gamma)}\big)\Big], \qquad (42)
\end{aligned}
$$

$$
n^{\xi\mu}={}_{\underline{0}}j_{\underline{1}}j_{\underline{0}}\lambda_{\underline{1}}\lambda\lambda L^{9}\overline{B}^{\xi\mu\alpha\beta}\Big[-\alpha\big(\overline{\varpi}_{\alpha}-
$$

$$-\lambda b^{(\alpha)}_{(\alpha)} w_{\alpha}\Big)\Big\|_\beta + \bar\alpha\Big(w_\alpha - \lambda b^{(\alpha)}_{(\alpha)} \bar w_\alpha\Big)\Big\|_\beta +$$

$$+\alpha b_{\alpha\beta}\Big(\bar w_3 - \lambda b^{(\alpha)}_{(\alpha)} w_3\Big) - \bar\alpha b_{\alpha\beta}\Big(w_3 - \lambda b^{(\alpha)}_{(\alpha)}\bar w_3\Big)+$$

$$+\lambda\Big(\alpha\bar\Upsilon^\gamma_{\alpha\beta} - \lambda\bar\alpha \Upsilon^\gamma_{\alpha\beta}\Big)\Big(\bar w_\gamma - \lambda b^{(\gamma)}_{(\gamma)} w_\gamma\Big)+$$

$$+\lambda\Big(-\bar\alpha\bar\Upsilon^\gamma_{\alpha\beta}+\lambda\alpha\Upsilon^\gamma_{\alpha\beta}\Big)\Big(w_\gamma - \lambda b^{(\gamma)}_{(\gamma)}\bar w_\gamma\Big)\Big]+$$

$$+{}_2 j_1 j_2 \lambda_1 \lambda\lambda L^9 B^{\xi\mu\alpha\beta}\Big[\bar\alpha\Big(\bar w_\alpha - \lambda b^{(\alpha)}_{(\alpha)} w_\alpha\Big)\Big\|_\beta -$$

$$-\alpha\Big(w_\alpha - \lambda b^{(\alpha)}_{(\alpha)}\bar w_\alpha\Big)\Big\|_\beta - \bar\alpha b_{\alpha\beta}\Big(\bar w_3 - \lambda b^{(\alpha)}_{(\alpha)} w_3\Big)+$$

$$+\alpha b_{\alpha\beta}\Big(w_3 - \lambda b^{(\alpha)}_{(\alpha)}\bar w_3\Big)+$$

$$+\lambda\Big(-\bar\alpha\bar\Upsilon^\gamma_{\alpha\beta}+\lambda\alpha\Upsilon^\gamma_{\alpha\beta}\Big)\Big(\bar w_\gamma - \lambda b^{(\gamma)}_{(\gamma)} w_\gamma\Big)+$$

$$+\lambda\Big(\alpha\bar\Upsilon^\gamma_{\alpha\beta}-\lambda\bar\alpha\Upsilon^\gamma_{\alpha\beta}\Big)\Big(w_\gamma - \lambda b^{(\gamma)}_{(\gamma)}\bar w_\gamma\Big)\Big]+$$

$$+\frac{1}{2}\,{}_2 j_1 j_2\lambda_1^2\lambda^2\lambda L^9 \bar B^{\xi\mu\alpha\beta}\Big[-\bar\beta\Big(\bar w_{3,\alpha}+$$

$$+\bar w_\alpha b^{(\alpha)}_{(\alpha)}\Big)\Big\|_\beta + \beta\Big(w_{3,\alpha}+w_\alpha b^{(\alpha)}_{(\alpha)}\Big)\Big\|_\beta +$$

$$+\lambda\Big(\bar\beta\bar\Upsilon^\gamma_{\alpha\beta}-\lambda\beta\Upsilon^\gamma_{\alpha\beta}\Big)\Big(\bar w_{3,\gamma}+\bar w_\gamma b^{(\gamma)}_{(\gamma)}\Big)$$

$$+\lambda\Big(-\beta\bar\Upsilon^\gamma_{\alpha\beta}+\lambda\bar\beta\Upsilon^\gamma_{\alpha\beta}\Big)\Big(w_{3,\gamma}+w_\gamma b^{(\gamma)}_{(\gamma)}\Big)\Big]+$$

$$+\frac{1}{2}\,{}_2 j_1 j_2\lambda_1^2\lambda^2\lambda L^9 B^{\xi\mu\alpha\beta}\Big[\beta\Big(\bar w_{3,\alpha}+$$

$$+\bar{w}_{\alpha} b^{(\alpha)}_{(\alpha)}\Big)_{\|\beta} - \bar{\beta}\Big(w_{3,\alpha} + w_{\alpha} b^{(\alpha)}_{(\alpha)}\Big)_{\|\beta} +$$

$$+\lambda\Big(-\beta\bar{\Upsilon}^{\gamma}_{\alpha\beta} + \lambda\bar{\beta}\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(\bar{w}_{3,\gamma} + \bar{w}_{\gamma} b^{(\gamma)}_{(\gamma)}\Big) +$$

$$+\lambda\Big(\bar{\beta}\bar{\Upsilon}^{\gamma}_{\alpha\beta} - \lambda\beta\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(w_{3,\gamma} + w_{\gamma} b^{(\gamma)}_{(\gamma)}\Big)\Big], \qquad (43)$$

$$\bar{m}^{\xi\mu} = \frac{1}{2}\, {}_{0}J_{1}J_{0}\lambda_{1}^{2}\lambda^{2}L^{9}\bar{B}^{\xi\mu\alpha\beta}\Big[-\beta\Big(\bar{w}_{\alpha} -$$

$$-\lambda b^{(\alpha)}_{(\alpha)} w_{\alpha}\Big)_{\|\beta} + \bar{\beta}\Big(w_{\alpha} - \lambda b^{(\alpha)}_{(\alpha)}\bar{w}_{\alpha}\Big)_{\|\beta} +$$

$$+\beta b_{\alpha\beta}\Big(\bar{w}_{3} - \lambda b^{(\alpha)}_{(\alpha)} w_{3}\Big) - \bar{\beta} b_{\alpha\beta}\Big(w_{3} - \lambda b^{(\alpha)}_{(\alpha)}\bar{w}_{3}\Big) +$$

$$+\lambda\Big(\beta\bar{\Upsilon}^{\gamma}_{\alpha\beta} - \lambda\bar{\beta}\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(\bar{w}_{\gamma} - \lambda b^{(\gamma)}_{(\gamma)} w_{\gamma}\Big) +$$

$$+\lambda\Big(-\bar{\beta}\bar{\Upsilon}^{\gamma}_{\alpha\beta} + \lambda\beta\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(w_{\gamma} - \lambda b^{(\gamma)}_{(\gamma)}\bar{w}_{\gamma}\Big)\Big] +$$

$$+\frac{1}{2}\, {}_{0}J_{1}J_{0}\lambda_{1}^{2}\lambda^{2}L^{9}B^{\xi\mu\alpha\beta}\Big[\bar{\beta}\Big(\bar{w}_{\alpha} -$$

$$-\lambda b^{(\alpha)}_{(\alpha)} w_{\alpha}\Big)_{\|\beta} - \beta\Big(w_{\alpha} - \lambda b^{(\alpha)}_{(\alpha)}\bar{w}_{\alpha}\Big)_{\|\beta} -$$

$$-\bar{\beta} b_{\alpha\beta}\Big(\bar{w}_{3} - \lambda b^{(\alpha)}_{(\alpha)} w_{3}\Big) + \beta b_{\alpha\beta}\Big(w_{3} - \lambda b^{(\alpha)}_{(\alpha)}\bar{w}_{3}\Big) +$$

$$+\lambda\Big(-\bar{\beta}\bar{\Upsilon}^{\gamma}_{\alpha\beta} + \lambda\beta\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(\bar{w}_{\gamma} - \lambda b^{(\gamma)}_{(\gamma)} w_{\gamma}\Big) +$$

$$+\lambda\Big(\beta\bar{\Upsilon}^{\gamma}_{\alpha\beta} - \lambda\bar{\beta}\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(w_{\gamma} - \lambda b^{(\gamma)}_{(\gamma)}\bar{w}_{\gamma}\Big)\Big] +$$

$$+\frac{1}{3}\, {}_{0}J_{1}J_{0}\lambda_{1}^{3}\lambda^{3}L^{9}\bar{B}^{\xi\mu\alpha\beta}\Big[-\bar{\gamma}\Big(\bar{w}_{3,\alpha} +$$

$$+\overline{w}_\alpha b^{(\alpha)}_{(\alpha)}\Big)\Big\|_\beta+\gamma\Big(w_{3,\alpha}+w_\alpha b^{(\alpha)}_{(\alpha)}\Big)\Big\|_\beta+$$

$$+\lambda\Big(\overline{\gamma}\overline{\Upsilon}^{\gamma}_{\alpha\beta}-\lambda\gamma\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(\overline{w}_{3,\gamma}+\overline{w}_\gamma b^{(\gamma)}_{(\gamma)}\Big)$$

$$+\lambda\Big(-\gamma\overline{\Upsilon}^{\gamma}_{\alpha\beta}+\lambda\overline{\gamma}\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(w_{3,\gamma}+w_\gamma b^{(\gamma)}_{(\gamma)}\Big)\Big]+$$

$$+\frac{1}{3}\,{}_0j_1j_0\lambda^3_1\lambda^3L^9B^{\xi\mu\alpha\beta}\Big[\gamma\Big(\overline{w}_{3,\alpha}+$$

$$+\overline{w}_\alpha b^{(\alpha)}_{(\alpha)}\Big)\Big\|_\beta-\overline{\gamma}\Big(w_{3,\alpha}+w_\alpha b^{(\alpha)}_{(\alpha)}\Big)\Big\|_\beta+$$

$$+\lambda\Big(-\gamma\overline{\Upsilon}^{\gamma}_{\alpha\beta}+\lambda\overline{\gamma}\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(\overline{w}_{3,\gamma}+\overline{w}_\gamma b^{(\gamma)}_{(\gamma)}\Big)+$$

$$+\lambda\Big(\overline{\gamma}\overline{\Upsilon}^{\gamma}_{\alpha\beta}-\lambda\gamma\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(w_{3,\gamma}+w_\gamma b^{(\gamma)}_{(\gamma)}\Big)\Big]\Big], \qquad (44)$$

$$m^{\xi\mu}=\frac{1}{2}\,{}_0j_1j_0\lambda^2_1\lambda^2\lambda L^9\overline{B}^{\xi\mu\alpha\beta}\Big[\overline{\beta}\Big(\overline{w}_\alpha-$$

$$-\lambda b^{(\alpha)}_{(\alpha)}w_\alpha\Big)\Big\|_\beta-\beta\Big(w_\alpha-\lambda b^{(\alpha)}_{(\alpha)}\overline{w}_\alpha\Big)\Big\|_\beta-$$

$$-\overline{\beta}b_{\alpha\beta}\Big(\overline{w}_3-\lambda b^{(\alpha)}_{(\alpha)}w_3\Big)+\beta b_{\alpha\beta}\Big(w_3-\lambda b^{(\alpha)}_{(\alpha)}\overline{w}_3\Big)+$$

$$+\lambda\Big(-\overline{\beta}\overline{\Upsilon}^{\gamma}_{\alpha\beta}+\lambda\beta\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(\overline{w}_\gamma-\lambda b^{(\gamma)}_{(\gamma)}w_\gamma\Big)+$$

$$+\lambda\Big(\beta\overline{\Upsilon}^{\gamma}_{\alpha\beta}-\lambda\overline{\beta}\Upsilon^{\gamma}_{\alpha\beta}\Big)\Big(w_\gamma-\lambda b^{(\gamma)}_{(\gamma)}\overline{w}_\gamma\Big)\Big]+$$

$$+\frac{1}{2}\,{}_0j_1j_0\lambda^2_1\lambda^2\lambda L^9B^{\xi\mu\alpha\beta}\Big[-\beta\Big(\overline{w}_\alpha-$$

$$-\lambda b^{(\alpha)}_{(\alpha)}w_\alpha\Big)\Big\|_\beta+\overline{\beta}\Big(w_\alpha-\lambda b^{(\alpha)}_{(\alpha)}\overline{w}_\alpha\Big)\Big\|_\beta+$$

$$
\begin{aligned}
&+\beta b_{\alpha\beta}\left(\overline{w_3}-\lambda b^{(\alpha)}_{(\alpha)}w_3\right)-\bar{\beta}\, b_{\alpha\beta}\left(w_3-\lambda b^{(\alpha)}_{(\alpha)}\overline{w_3}\right)+\\
&+\lambda\left(\beta\overline{\Upsilon}{}^{\gamma}_{\alpha\beta}-\lambda\bar{\beta}\,\Upsilon^{\gamma}_{\alpha\beta}\right)\left(\overline{w_\gamma}-\lambda b^{(\gamma)}_{(\gamma)}w_\gamma\right)+\\
&+\lambda\left(-\bar{\beta}\,\overline{\Upsilon}{}^{\gamma}_{\alpha\beta}+\lambda\beta\Upsilon^{\gamma}_{\alpha\beta}\right)\left(w_\gamma-\lambda b^{(\gamma)}_{(\gamma)}\overline{w_\gamma}\right)\Big]+\\
&+\tfrac{1}{3}\,{}_{0}j_{1}j_{0}\lambda_{1}^{3}\lambda^{3}\lambda L^{9}\,\bar{B}^{\xi\mu\alpha\beta}\Big[\gamma\left(\overline{w_{3,\alpha}}+\right.\\
&\left.+w_\alpha b^{(\alpha)}_{(\alpha)}\right)_{\|\beta}-\bar{\gamma}\left(w_{3,\alpha}+w_\alpha b^{(\alpha)}_{(\alpha)}\right)_{\|\beta}+\\
&+\lambda\left(-\gamma\overline{\Upsilon}{}^{\gamma}_{\alpha\beta}+\lambda\bar{\gamma}\,\Upsilon^{\gamma}_{\alpha\beta}\right)\left(\overline{w_{3,\gamma}}+\overline{w_\gamma}\, b^{(\gamma)}_{(\gamma)}\right)+\\
&+\lambda\left(\bar{\gamma}\,\overline{\Upsilon}{}^{\gamma}_{\alpha\beta}-\lambda\gamma\Upsilon^{\gamma}_{\alpha\beta}\right)\left(w_{3,\gamma}+w_\gamma b^{(\gamma)}_{(\gamma)}\right)\Big]+\\
&+\tfrac{1}{3}\,{}_{0}j_{1}j_{0}\lambda_{1}^{3}\lambda^{3}\lambda L^{9}B^{\xi\mu\alpha\beta}\Big[-\bar{\gamma}\left(\overline{w_{3,\alpha}}+\right.\\
&\left.+\overline{w_\alpha}\, b^{(\alpha)}_{(\alpha)}\right)_{\|\beta}+\gamma\left(w_{3,\alpha}+w_\alpha b^{(\alpha)}_{(\alpha)}\right)_{\|\beta}+\\
&+\lambda\left(\bar{\gamma}\,\overline{\Upsilon}{}^{\gamma}_{\alpha\beta}-\lambda\gamma\Upsilon^{\gamma}_{\alpha\beta}\right)\left(\overline{w_{3,\gamma}}+\overline{w_\gamma}\, b^{(\gamma)}_{(\gamma)}\right)+\\
&+\lambda\left(-\gamma\overline{\Upsilon}{}^{\gamma}_{\alpha\beta}+\lambda\bar{\gamma}\,\Upsilon^{\gamma}_{\alpha\beta}\right)\left(w_{3,\gamma}+w_\gamma b^{(\gamma)}_{(\gamma)}\right)\Big].\qquad(45)
\end{aligned}
$$

Remembering that ${}_{\underline{n}}\lambda$ and λ^2 have been neglected when compared to one, care must be taken when using (42), (43), (44) and (45) since ${}_{\underline{n}}\lambda$ and λ are contained in $\bar{\alpha}, \alpha, \bar{\beta}, \beta, \bar{\gamma}$ and γ and λ may be contained in $B^{\alpha\beta\gamma\eta}$.

Terms multiplied by α, β and γ when ${}_{\underline{o}}\lambda = {}_{\underline{1}}\lambda$, by $B^{\alpha\beta\gamma\eta}$ when the facings have the same physical properties, by $\lambda b^{(\alpha)}_{(\alpha)}$ and by $\bar{\Upsilon}^{\gamma}_{\alpha\beta}$ and $\Upsilon^{\gamma}_{\alpha\beta}$ are due to the variation in the geometry thru the composite shell thickness. If the sandwich shell is thin these terms can be neglected. Hence for a thin sandwich shell with equal facings, (42) reduces to

$$\bar{n}^{\xi\mu} = {}_{\underline{o}}\lambda_{\underline{1}}\lambda L^{9} \bar{B}^{\xi\mu\alpha\beta}\left[\bar{\alpha}\left(\bar{w}_{\alpha\|\beta} - b_{\alpha\beta}\bar{w}_{3}\right)\right] - $$
$$-\frac{1}{2}\,{}_{\underline{o}}\lambda^{2}_{\underline{1}}\lambda^{2} L^{9} \bar{B}^{\xi\mu\alpha\beta}\left[\bar{\beta}\left(w_{3,\alpha} + w_{\alpha} b^{(\alpha)}_{(\alpha)}\right)_{\|\beta}\right].$$

This is the same stress resultant-displacement relation obtained in [11].

Equations (9), (10), (38), (39), (40), (41), (42), (43), (44) and (45) form a system of 21 simultaneous differential equations in the 21 variables $\bar{w}_r, w_r, \sigma^{33}, \bar{s}^{\alpha}, \bar{n}^{\alpha\beta}, n^{\alpha\beta}, \bar{m}^{\alpha\beta}$ and $m^{\alpha\beta}$.

11. Boundary Conditions

The boundary condition for the edge of the core has already been given, i.e. the shear resultant on the edge of the core,

$$\mathscr{S} = u_{\alpha}\bar{s}^{\alpha},$$

must be specified.

Since on a normal to the core mid-surface at the edge of the composite shell stresses may be prescribed for one facing while displacements are prescribed for the other facing, the boundary conditions for the individual facings will be given. Using [3]

$$\frac{\partial}{\partial\theta^\alpha} = \left({}_{\underline{n}}u_\alpha \frac{\partial}{\partial_{\underline{n}} n} + {}_{\underline{n}}t_\alpha \frac{\partial}{\partial_{\underline{n}} s}\right),$$

integrating by parts and then setting the resulting coefficients of the varied quantities in the line integrals equal to zero, the required facing boundary conditions are

$$\left[{}_{\underline{n}}n^{\alpha\beta}\left(1 \mp \lambda b^{(\alpha)}_{(\alpha)}\right) - {}_{\underline{n}}m^{\alpha\beta} b^{(\alpha)}_{(\alpha)}\right]{}_{\underline{n}}u_\beta =$$

$$= \left[{}_{\underline{n}}\tilde{n}^{\alpha\beta}\left(1 \mp \lambda b^{(\alpha)}_{(\alpha)}\right) - {}_{\underline{n}}\tilde{m}^{\alpha\beta} b^{(\alpha)}_{(\alpha)}\right]{}_{\underline{n}}u_\beta ,$$

$${}_{\underline{n}}m^{\alpha\beta}{}_{\underline{n}}u_\alpha {}_{\underline{n}}u_\beta = {}_{\underline{n}}\tilde{m}^{\alpha\beta}{}_{\underline{n}}u_\alpha {}_{\underline{n}}u_\beta \qquad (46)$$

and

$$\frac{\partial}{\partial_{\underline{n}} s}\left[\frac{{}_{\underline{n}}t_\alpha {}_{\underline{n}}u_\beta}{{}_{\underline{n}}j}{}_{\underline{n}}m^{\alpha\beta}\right] + \frac{{}_{\underline{n}}u_\alpha}{{}_{\underline{n}}j}{}_{\underline{n}}q^\alpha =$$

$$= \frac{\partial}{\partial_{\underline{n}} s}\left[\frac{{}_{\underline{n}}t_\alpha {}_{\underline{n}}u_\beta}{{}_{\underline{n}}j}{}_{\underline{n}}\tilde{m}^{\alpha\beta}\right] + \frac{{}_{\underline{n}}u_\alpha}{{}_{\underline{n}}j}{}_{\underline{n}}\tilde{q}^\alpha$$

on ${}_{\underline{n}}C_1$ and

$${}_{\underline{n}}V_\beta = {}_{\underline{n}}\tilde{V}_\beta ,$$

$${}_{\underline{n}}V_3 = {}_{\underline{n}}\tilde{V}_3$$

and

$${}_{\underline{n}}V_{3,\alpha} = {}_{\underline{n}}\tilde{V}_{3,\alpha}$$

on ${}_{\underline{n}}C_2$.

These boundary conditions have the same form as those obtained in [4].

12. Stress-Strain Relations for a Sandwich Shell with a Viscoelastic Core

In the following two sections a sandwich shell with a viscoelastic core is considered. Representative equations for this shell are displayed.

Only the core will be presumed viscoelastic, however, viscoelastic facings could be treated in the same way.

The core stress-strain relations are altered as follows; E and E^{α} are replaced by $\eta^{*}E$ and $\eta^{*}E^{\alpha}$, respectively, and all other functions of time are replaced by their Laplace transforms, e.g. (9) becomes

$$
{}^{*}w_{3} = \frac{\lambda L}{2\eta^{*}E}\Big[{}^{*}\sigma^{33} + \frac{2\lambda}{3L^{2}}\,{}^{*}s^{\alpha}\|_{\alpha}\big(b^{(\alpha)}_{(\alpha)} - h\big) + \frac{2\lambda}{3L^{2}}\,{}^{*}s^{\alpha}b^{(\alpha)}_{(\alpha),\alpha}\Big].
$$

Since $\bar{B}^{\alpha\beta\gamma\eta}$ and $B^{\alpha\beta\gamma\eta}$ are not functions of time, the stress resultant-displacement relations for the composite shell are converted simply by substituting Laplace transforms for all time functions. To illustrate this a few terms of the equation corresponding to (42) are presented;

$$
\begin{aligned}
{}^{*}\bar{n}^{\xi\mu} = {}_{\underline{2}}j_{\underline{1}}j_{\underline{2}}\lambda_{\underline{1}}\lambda L^{9}\bar{B}^{\xi\mu\alpha\beta}\Big[&\bar{\alpha}\big({}^{*}\bar{w}_{\alpha} - \lambda b^{(\alpha)}_{(\alpha)}\,{}^{*}w_{\alpha}\big)\|_{\beta} - \alpha\big({}^{*}w_{\alpha} - \lambda b^{(\alpha)}_{(\alpha)}\,{}^{*}\bar{w}_{\alpha}\big)\|_{\beta} - \\
&- \bar{\alpha}b_{\alpha\beta}\big({}^{*}\bar{w}_{3} - \lambda b^{(\alpha)}_{(\alpha)}\,{}^{*}w_{3}\big) + \alpha b_{\alpha\beta}\big({}^{*}w_{3} - \lambda b^{(\alpha)}_{(\alpha)}\,{}^{*}\bar{w}_{3}\big) + \\
&+ \lambda\big(-\bar{\alpha}\bar{\Gamma}^{\gamma}_{\alpha\beta} + \lambda\alpha\Gamma^{\gamma}_{\alpha\beta}\big)\big({}^{*}\bar{w}_{\gamma} - \lambda b^{(\gamma)}_{(\gamma)}\,{}^{*}w_{\gamma}\big) +
\end{aligned}
$$

$$+\lambda\left(\alpha\bar{\Upsilon}^{\gamma}_{\alpha\beta}-\lambda\bar{\alpha}\Upsilon^{\gamma}_{\alpha\beta}\right)\left(\overset{*}{w}_{\gamma}-\lambda b^{(\gamma)}_{(\gamma)}\overset{*}{\bar{w}}_{\gamma}\right)\Big]+$$

$$+ - - - .$$

13. Equilibrium Equations and Boundary Conditions for a Sandwich Shell with a Viscoelastic Core

The equilibrium equations for the composite shell and the boundary conditions for the individual facings and the core are obtained by merely replacing all functions of time by their Laplace transforms, e.g. (38) and (46) become

$$\overset{*}{\bar{p}}{}^{3}+\overset{*}{\bar{s}}{}^{\alpha}\|_{\alpha}+b_{\alpha\beta}\overset{*}{\bar{n}}{}^{\alpha\beta}+\overset{*}{\bar{m}}{}^{\alpha\beta}\|_{\alpha\beta}-b^{(\alpha)}_{(\alpha)}b_{\alpha\beta}\overset{*}{n}{}^{\alpha\beta}+$$

$$+\lambda\left(\bar{\Upsilon}^{\gamma}_{\alpha\beta}\overset{*}{\bar{m}}{}^{\alpha\beta}+\Upsilon^{\gamma}_{\alpha\beta}\overset{*}{m}{}^{\alpha\beta}\right)\Big\|_{\gamma}=0$$

and

$${}_{\underline{n}}\overset{*}{m}{}^{\alpha\beta}\,{}_{\underline{n}}u_{\alpha}\,{}_{\underline{n}}u_{\beta}={}_{\underline{n}}\overset{*}{\tilde{m}}{}^{\alpha\beta}\,{}_{\underline{n}}u_{\alpha}\,{}_{\underline{n}}u_{\beta}$$

It has been assumed that σ_{α} and ${}_{\underline{n}}C_{\alpha}$ are independent of time.

EXAMPLES

The theory presented here is valid for sandwich shells (plates) with thin Kirchhoff-Love shell (plate) facings. However, the following three examples are only concerned with sandwich shells (plates) with membrane facings. The facings are presumed membranes so that the influence of a hole, of orthotropic facings and of a viscoelastic core on the behavior of a sandwich shell (plate) can be studied without unduly complicating the examples.

14. Circular Plate with a Circular Hole at the Center

Consider a simply supported circular plate with a circular hole at the center loaded by a uniformly distributed bending couple around the outer boundary. The facings are isotropic membranes with similar physical properties and equal thicknesses. The core is presumed isotropic. The dimensionless surface coordinates are

$$\theta^1 = \frac{\rho}{R} \;, \quad \theta^2 = \phi \;,$$

where ρ and ϕ are polar coordinates (see figure 2).

Due to the symmetry of the plate and the applied edge couple,

$$\bar{s}^2 = \bar{w}_2 = w_2 = \bar{n}^{12} = n^{12} = 0$$

and the remaining dependent variables are independent of θ^2.

The equilibrium equations which are not identically satisfied are

$$\bar{s}^1{}_{,1} + \frac{\bar{s}^1}{\theta^1} = 0 \;, \tag{47}$$

$$\sigma^{33} = 0 \;, \tag{48}$$

$$\bar{n}^{11}{}_{,1} + \frac{\bar{n}^{11}}{\theta^1} - \theta^1 \bar{n}^{22} = 0 \;, \tag{49}$$

$$n^{11}{}_{,1} + \frac{n^{11}}{\theta^1} - \theta^1 n^{22} - \bar{s}^1 = 0 . \tag{50}$$

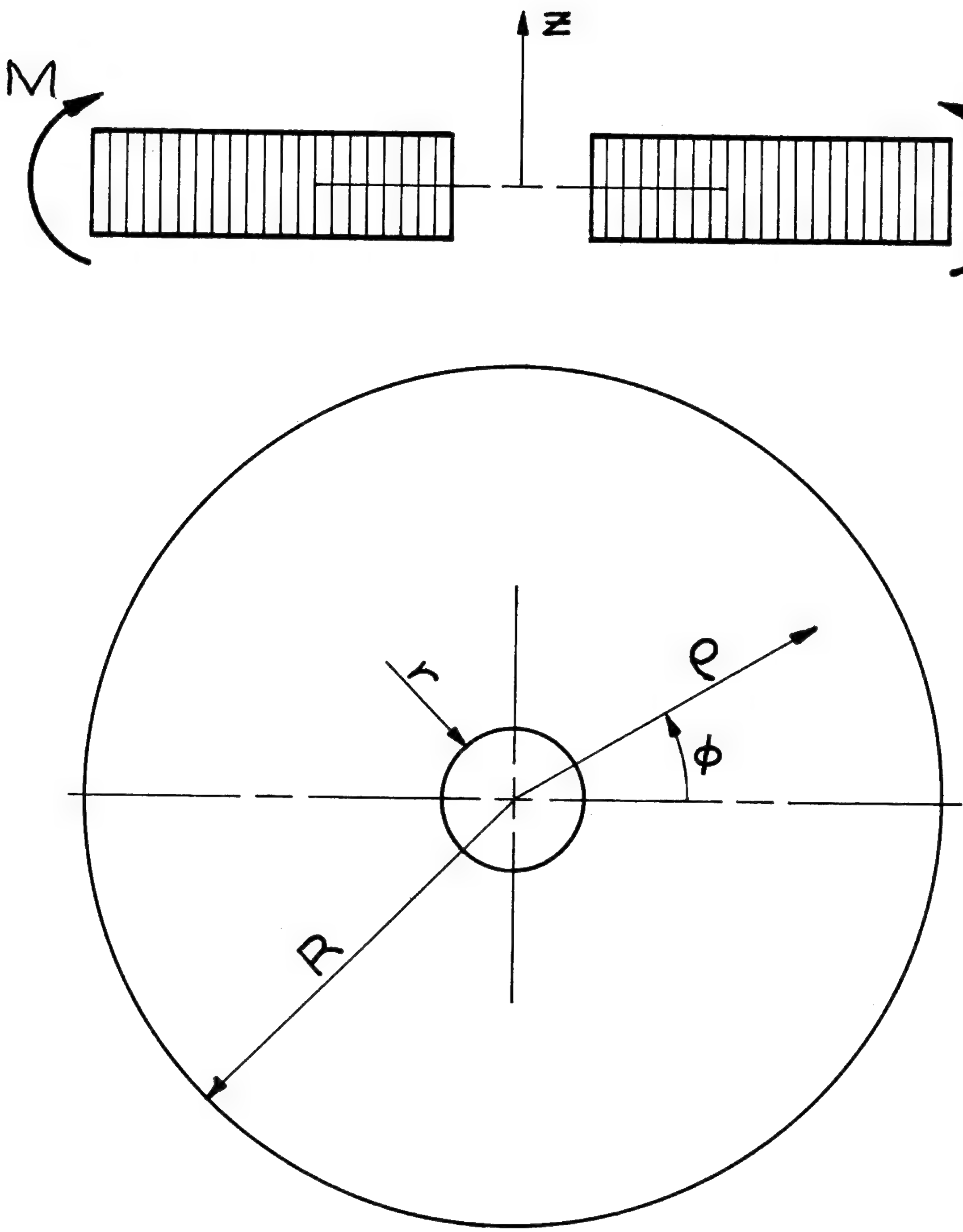

FIG. 2, CIRCULAR PLATE

The core stress-strain relations are

$$w_3 = \frac{\lambda L}{2E}\sigma^{33}, \tag{51}$$

$$w_1 = -\lambda \bar{w}_{3,1} + \frac{\bar{s}^1}{2LG} - \frac{\lambda^2}{6LE}\bar{s}^{\beta}\|_{\beta 1}\,. \tag{52}$$

The stress resultant-displacement relations for the composite plate are

$$\bar{n}^{11} = {}_{0}\lambda R \frac{{}_{0}E}{1-\nu^2}\Big[2\Big(\bar{w}_{1,1} + \frac{\nu}{\theta^1} w_1\Big) - {}_{0}\lambda\Big(w_{3,11} + \frac{\nu}{\theta^1} w_{3,1}\Big)\Big] \tag{53}$$

$$\bar{n}^{22} = \frac{{}_{0}\lambda R}{(\theta^1)^2}\frac{{}_{0}E}{1-\nu^2}\Big[2\Big(\nu\bar{w}_{1,1} + \frac{1}{\theta^1}\bar{w}_1\Big) - {}_{0}\lambda\Big(\nu w_{3,11} + \frac{1}{\theta^1} w_{3,1}\Big)\Big] \tag{54}$$

$$n^{11} = \lambda\,{}_{0}\lambda R \frac{{}_{0}E}{1-\nu^2}\Big[2\Big(w_{1,1} + \frac{\nu}{\theta^1} w_1\Big) - {}_{0}\lambda\Big(\bar{w}_{3,11} + \frac{\nu}{\theta^1}\bar{w}_{3,1}\Big)\Big] \tag{55}$$

$$n^{22} = \frac{\lambda\,{}_{0}\lambda R}{(\theta^1)^2}\frac{{}_{0}E}{1-\nu^2}\Big[2\Big(\nu w_{1,1} + \frac{1}{\theta^1} w_1\Big) - {}_{0}\lambda\Big(\nu\bar{w}_{3,11} + \frac{1}{\theta^1}\bar{w}_{3,1}\Big)\Big]. \tag{56}$$

The boundary conditions are

$$\big[\bar{s}^1\big]_{\theta^1=1} = 0, \tag{57}$$

$$\big[\bar{n}^{11}\big]_{\theta^1=1} = 0,\quad \big[\bar{n}^{11}\big]_{\theta^1=r/R} = 0, \tag{58}$$

$$\big[n^{11}\big]_{\theta^1=1} = -\frac{M}{R},\quad \big[n^{11}\big]_{\theta^1=r/R} = 0, \tag{59}$$

$$\big[\bar{w}_3\big]_{\theta^1=1} = 0. \tag{60}$$

From (48) and (51) one sees that $w_3 = 0.$ (61)

The solution of (47) and (57) is

$$\bar{s}^1 = 0. \tag{62}$$

Substituting (53), (54) and (61) into (49) gives

$$\left[\frac{1}{\theta^1}\left(\theta^1 \bar{w}_1\right)_{,1}\right]_{,1} = 0. \tag{63}$$

Equation (63) and the boundary conditions (58) yield

$$\bar{w}_1 = 0 \tag{64}$$

Hence,

$$\bar{n}^{11} = \bar{n}^{22} = 0.$$

In the same way (64) was obtained, from (50), (62), (52), (55), (56), (59) and (60) one finds

$$\bar{w}_3 = \left[\frac{M}{2\lambda\,{}_o\lambda(2\lambda + {}_o\lambda)(R^2 - r^2)R^2_o E}\right] \cdot$$
$$\cdot \left\{(1-\nu)R^2\left[(\theta^1)^2 - 1\right] + 2(1+\nu)r^2 \log(\theta^1)\right\}$$

This solution has exactly the same character as the solution of a homogeneous plate [12]. If the facings had been thin plates instead of membranes the problem would have been greatly complicated and the character of the solution would have been different. The character of the solution would depend on the boundary conditions, however, w_3 and $\bar{s}^1$ in general would not be zero and $\bar{w}_3$ would be considerably more complicated.

A sandwich plate with equal facings and a hole (circular or not)

loaded by the same inplane edge tensions on each facing has exactly the same solution as a homogeneous plate. In this case the facings can be either membranes or thin plates.

15. Square Plate with Orthotropic Facings

To illustrate the influence of anisotropic facings consider a simply supported square plate with orthotropic membrane facings. The principal axes of each facing are parallel to the coordinate axes. The facings have the same thickness and the core is isotropic. A uniform transverse load is applied to the upper facing.

The dimensionless surface coordinates are

$$\theta^1 = \frac{x}{L}, \quad \theta^2 = \frac{y}{L}$$

(see figure 3) and it is assumed that

$$\lambda = \frac{1}{100}, \quad {}_0\lambda \frac{1}{2000},$$

$$\frac{G}{{}_0E_1} = \frac{1}{100}, \quad \nu = \frac{3}{10}.$$

For this example ${}_nE_\alpha$ is the elastic modulus in the θ^α direction and ${}_nE_{12}$ is the cross modulus for the facings. If the facings are isotropic ${}_nE_\alpha$ and ${}_nE_{12}$ are equal to Young's modulus. It will be assumed that ${}_0E_1$ is the largest of all the moduli. For brevity it has been assumed that

$$\frac{{}_0E_2}{{}_0E_1} = \alpha_1, \quad \frac{{}_0E_{12}}{{}_0E_1} = \alpha_2,$$

$$\frac{{}_1E_1}{{}_0E_1} = \beta_1, \quad \frac{{}_1E_2}{{}_0E_1} = \beta_2, \quad \frac{{}_1E_{12}}{{}_0E_1} = \beta_3.$$

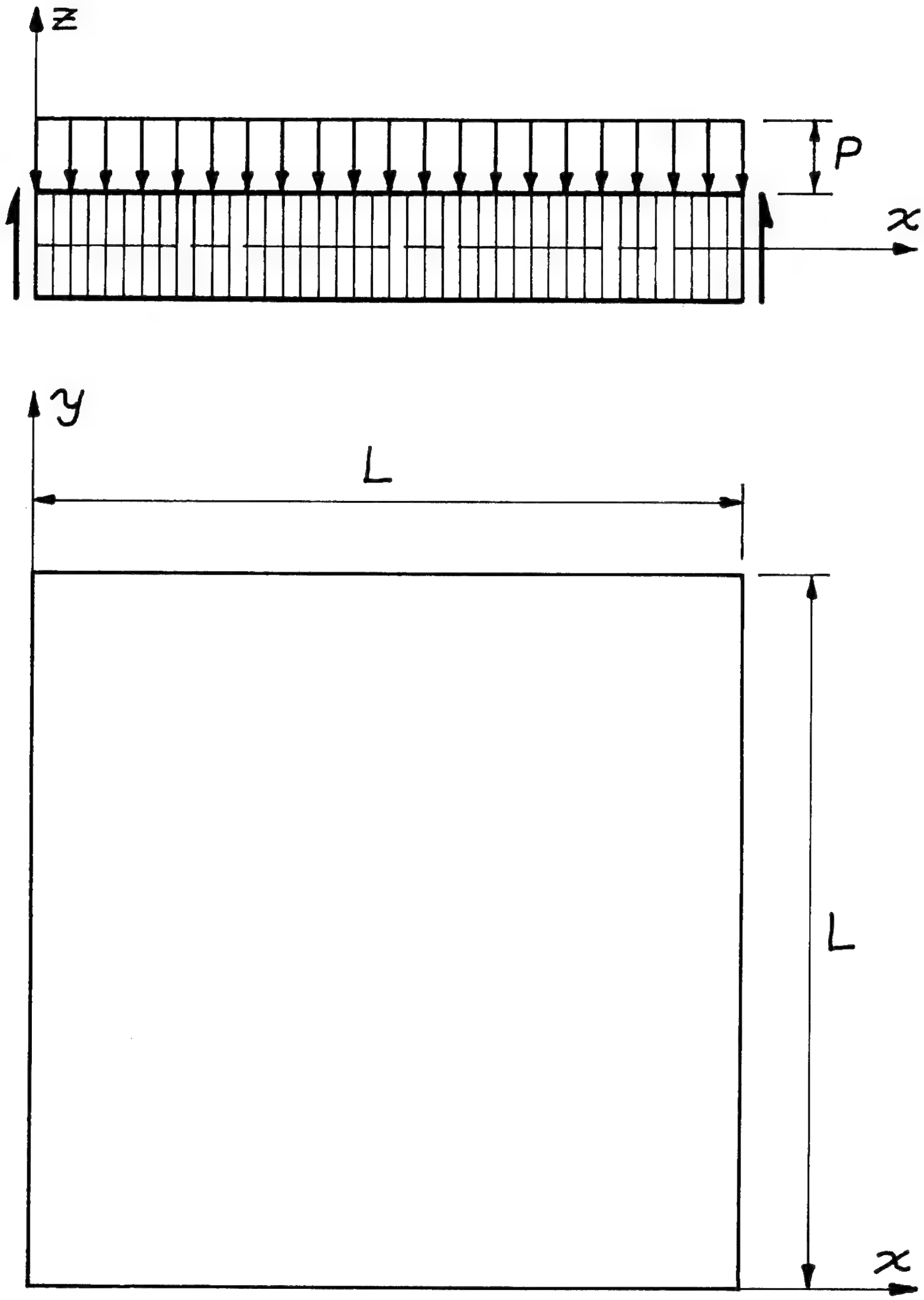

FIG. 3, SQUARE PLATE

Since the plate is simply supported and symmetric

$$\left.\begin{aligned} \bar{n}^{\alpha\beta} &= 0 \\ n^{12} &= 0 \end{aligned}\right\}. \tag{65}$$

The equilibrium equations not identically satisfied are

$$\bar{s}^{\alpha}{}_{,\alpha} - p = 0, \tag{66}$$

$$L^{2}\sigma^{33} + p = 0, \tag{67}$$

$$n^{\alpha\beta}{}_{,\beta} - \bar{s}^{\alpha} = 0. \tag{68}$$

The core stress-strain relations are

$$w_{3} = \frac{L}{200E}\sigma^{33}, \tag{69}$$

$$w_{\alpha} = -\frac{\bar{w}_{3,\alpha}}{100} + \frac{\bar{s}^{\alpha}}{2LG} - \frac{10^{-4}}{6LE}\bar{s}^{\beta}{}_{,\beta\alpha}. \tag{70}$$

The stress resultant-displacement relations for the composite plate are

$$\begin{aligned} \bar{n}^{\gamma\eta} = \frac{L^{5}}{10^{3}}\Big[&\bar{B}^{\gamma\eta\alpha\beta}\bar{w}_{\alpha,\beta} + B^{\gamma\eta\alpha\beta}w_{\alpha,\beta} - \\ &- \frac{1}{4\times 10^{3}}\left(\bar{B}^{\gamma\eta\alpha\beta}w_{3,\alpha\beta} + \bar{B}^{\gamma\eta\alpha\beta}\bar{w}_{3,\alpha\beta}\right)\Big], \end{aligned} \tag{71}$$

$$\begin{aligned} n^{\gamma\eta} = \frac{L^{5}}{10^{5}}\Big[&\bar{B}^{\gamma\eta\alpha\beta}w_{\alpha,\beta} + B^{\gamma\eta\alpha\beta}\bar{w}_{\alpha,\beta} - \\ &- \frac{1}{4\times 10^{3}}\left(\bar{B}^{\gamma\eta\alpha\beta}\bar{w}_{3,\alpha\beta} + B^{\gamma\eta\alpha\beta}w_{3,\alpha\beta}\right)\Big]. \end{aligned} \tag{72}$$

The boundary conditions are

$$\left[n^{(\alpha\alpha)}\right]_{\theta^{\alpha}=0,1} = 0, \quad \left[\bar{w}_{3}\right]_{\theta^{\alpha}=0,1} = 0. \tag{73}$$

From (67) it is seen that σ^{33} is a constant, hence, from (69) it follows that w_3 is a constant. According to (66), $\bar{S}^{\alpha}{}_{,\alpha}$ is a constant so that (70) reduces to

$$w_{\alpha} = -\frac{\bar{w}_{3,\alpha}}{100} + \frac{\bar{S}^{\alpha}}{2LG} \tag{74}$$

Using (65), (71) and (72) in (68) one obtains

$$\frac{41}{4000}\left[\bar{F}^{1111}\bar{w}_{3,111} + \bar{F}^{1122}\bar{w}_{3,122}\right] -$$

$$-\frac{1}{2LG}\left[\bar{F}^{1111}\bar{S}^{1}{}_{,11} + \bar{F}^{1122}\bar{S}^{2}{}_{,12}\right] +$$

$$+\frac{10^5}{L^5}\bar{S}^{1} = 0, \tag{75}$$

$$\frac{41}{4000}\left[\bar{F}^{2211}\bar{w}_{3,112} + \bar{F}^{2222}\bar{w}_{3,222}\right] -$$

$$-\frac{1}{2LG}\left[\bar{F}^{2211}\bar{S}^{1}{}_{,12} + \bar{F}^{2222}\bar{S}^{2}{}_{,22}\right] +$$

$$+\frac{10^5}{L^5}\bar{S}^{2} = 0, \tag{76}$$

where

$$\bar{F}^{(\alpha\alpha)(\beta\beta)} = \bar{B}^{(\alpha\alpha)(\beta\beta)} -$$

$$-B^{(\alpha\alpha)11}\left[\frac{\bar{B}^{2222}B^{11(\beta\beta)} - \bar{B}^{1122}B^{22(\beta\beta)}}{\bar{B}^{1111}\bar{B}^{2222} - (\bar{B}^{1122})^2}\right] -$$

$$-B^{(\alpha\alpha)22}\left[\frac{\bar{B}^{1111}B^{22(\beta\beta)} - \bar{B}^{1122}B^{11(\beta\beta)}}{\bar{B}^{1111}\bar{B}^{2222} - (\bar{B}^{1122})^2}\right].$$

Equations (66), (75) and (76) are three simultaneous differential equations in the three dependent variables $\bar{w}_3$, $\bar{S}^1$ and $\bar{S}^2$.

The following series satisfy the boundary conditions

$$\frac{\bar{w}_3}{LQ} = \sum_{r \,\&\, s = odd}^{\infty} A_{rs} \sin(r\pi\theta^1) \sin(s\pi\theta^2) ,$$

$$\frac{\bar{S}^1}{P} = \sum_{r \,\&\, s = odd}^{\infty} B_{rs} \cos(r\pi\theta^1) \sin(s\pi\theta^2) ,$$

$$\frac{\bar{S}^2}{P} = \sum_{r \,\&\, s = odd}^{\infty} C_{rs} \sin(r\pi\theta^1) \cos(s\pi\theta^2) ,$$

where

$$Q = \frac{P}{L_o^2 E_1} .$$

Substituting these series into (66), (75) and (76) one obtains three simultaneous algebraic equations in the three sets of constants A_{rs}, B_{rs} and C_{rs}. Solving these equations one finds

$$A_{rs} = -\left[\frac{16\times 10^5}{41\pi^4 rs}\right] \frac{\left\{\begin{aligned} & s^2(\Gamma_2+\Gamma_3)\left[2000 + r^2\pi^2(\Gamma_1-\Gamma_3)\right] + \\ & + r^2(\Gamma_1+\Gamma_3)\left[2000 + s^2\pi^2(\Gamma_2-\Gamma_3)\right] + \\ & + \frac{2000}{\pi^2}\left[4000 + r^2\pi(\Gamma_1-\Gamma_3) + \right. \\ & \qquad \left. + s^2\pi^2(\Gamma_2-\Gamma_3)\right] \end{aligned}\right\}}{\left\{\begin{aligned} & r^2(r^2\Gamma_1 + s^2\Gamma_3)\left[2000 + s^2\pi^2(\Gamma_2-\Gamma_3)\right] + \\ & + s^2(s^2\Gamma_2 + r^2\Gamma_3)\left[2000 + r^2\pi^2(\Gamma_1-\Gamma_3)\right] \end{aligned}\right\}},$$

$$B_{rs} = \frac{16\Gamma_3}{s\pi\left[2000 + r^2\pi^2(\Gamma_1 - \Gamma_3)\right]} + \frac{41\pi^3}{2\times10^5}\left[\frac{r^2\Gamma_1 + s^2\Gamma_3}{2000 + r^2\pi^2(\Gamma_1 - \Gamma_3)}\right] rA_{rs} ,$$

$$C_{rs} = \frac{16\Gamma_3}{r\pi\left[2000 + s^2\pi^2(\Gamma_2 - \Gamma_3)\right]} + \frac{41\pi^3}{2\times10^5}\left[\frac{s^2\Gamma_2 + r^2\Gamma_3}{2000 + s^2\pi^2(\Gamma_2 - \Gamma_3)}\right] sA_{rs} ,$$

where

$$\Gamma_1 = \frac{50}{91}\left(1+\beta_1\right) - \frac{50\left[100\left(1-\beta_1\right)^2\left(\alpha_1+\beta_2\right)-18\left(1-\beta_1\right)\left(\alpha_2+\beta_3\right)\left(\alpha_2-\beta_3\right)+9\left(1+\beta_2\right)\left(\alpha_2-\beta_3\right)^2\right]}{91\left[100\left(1+\beta_1\right)\left(\alpha_1+\beta_2\right)-9\left(\alpha_2+\beta_3\right)^2\right]},$$

$$\Gamma_2 = \frac{50}{91}\left(\alpha_1+\beta_2\right) - \frac{50\left[100\left(\alpha_1-\beta_2\right)^2\left(1+\beta_1\right)-18\left(\alpha_1-\beta_2\right)\left(\alpha_2+\beta_3\right)\left(\alpha_2-\beta_3\right)+9\left(\alpha_1+\beta_2\right)\left(\alpha_2-\beta_3\right)^2\right]}{91\left[100\left(1+\beta_1\right)\left(\alpha_1+\beta_2\right)-9\left(\alpha_2+\beta_3\right)^2\right]},$$

$$\Gamma_3 = \frac{15}{91}\left(\alpha_2 + \beta_3\right) -$$

$$- \frac{15\left[100\left(1-\beta_1\right)\left(\alpha_1+\beta_2\right)\left(\alpha_2-\beta_3\right) + 100\left(1+\beta_1\right)\left(\alpha_1-\beta_2\right)\left(\alpha_2-\beta_3\right) - \atop - 100\left(1-\beta_1\right)\left(\alpha_1-\beta_2\right)\left(\alpha_2+\beta_3\right) - 9\left(\alpha_2+\beta_3\right)\left(\alpha_2-\beta_3\right)^2\right]}{91\left[100\left(1+\beta_1\right)\left(\alpha_1+\beta_2\right) - 9\left(\alpha_2+\beta_3\right)^2\right]} .$$

The remaining unknown functions can now be determined. One obtains w_α from (74) and $n^{(\alpha\alpha)}$ from (68) and (73);

$$\frac{w_1}{LQ} = -\frac{1}{100}\sum_{r \not\leq s=odd}^{\infty} r\pi A_{rs}\cos(r\pi\theta^1)\sin(s\pi\theta^2) +$$

$$+ 50\sum_{r \not\leq s=odd}^{\infty} B_{rs}\cos(r\pi\theta^1)\sin(s\pi\theta^2),$$

$$\frac{w_2}{LQ} = -\frac{1}{100}\sum_{r \not\leq s=odd}^{\infty} s\pi A_{rs}\sin(r\pi\theta^1)\cos(s\pi\theta^2) +$$

$$+ 50\sum_{r \not\leq s=odd}^{\infty} C_{rs}\sin(r\pi\theta^1)\cos(s\pi\theta^2),$$

$$\frac{n^{11}}{P} = \sum_{r \not\leq s=odd}^{\infty} B_{rs}\left(\frac{1}{r\pi}\right)\sin(r\pi\theta^1)\sin(s\pi\theta^2),$$

$$\frac{n^{22}}{P} = \sum_{r \not\leq s=odd}^{\infty} C_{rs}\left(\frac{1}{s\pi}\right)\sin(r\pi\theta^1)\sin(s\pi\theta^2).$$

Figures 4, 5, 6 and 7 show displacements and stress resultants for a sandwich plate whose upper facing remains isotropic while its lower facing ranges over various degrees of orthotropy. ${}_{\perp}E_1$ is equal to Young's modulus for the upper facing and ${}_{\perp}E_2$ decreases from ${}_{\perp}E_1$. The cross modulus is assumed to have the following form

$${}_{\perp}E_{12} = \sqrt{{}_{\perp}E_1\,{}_{\perp}E_2}\ .$$

From the figures we see that as the lower facing ranges over various degrees of orthotropy all displacements and stress resultants behave as one would expect. The stress resultants $\overline{s}^1$ and n^{11} are larger than $\overline{s}^2$ and n^{22} since the plate stiffness in the θ^1 direction is greater than the stiffness in the θ^2 direction. For the same reason the rotation ω_1 is less than ω_2.

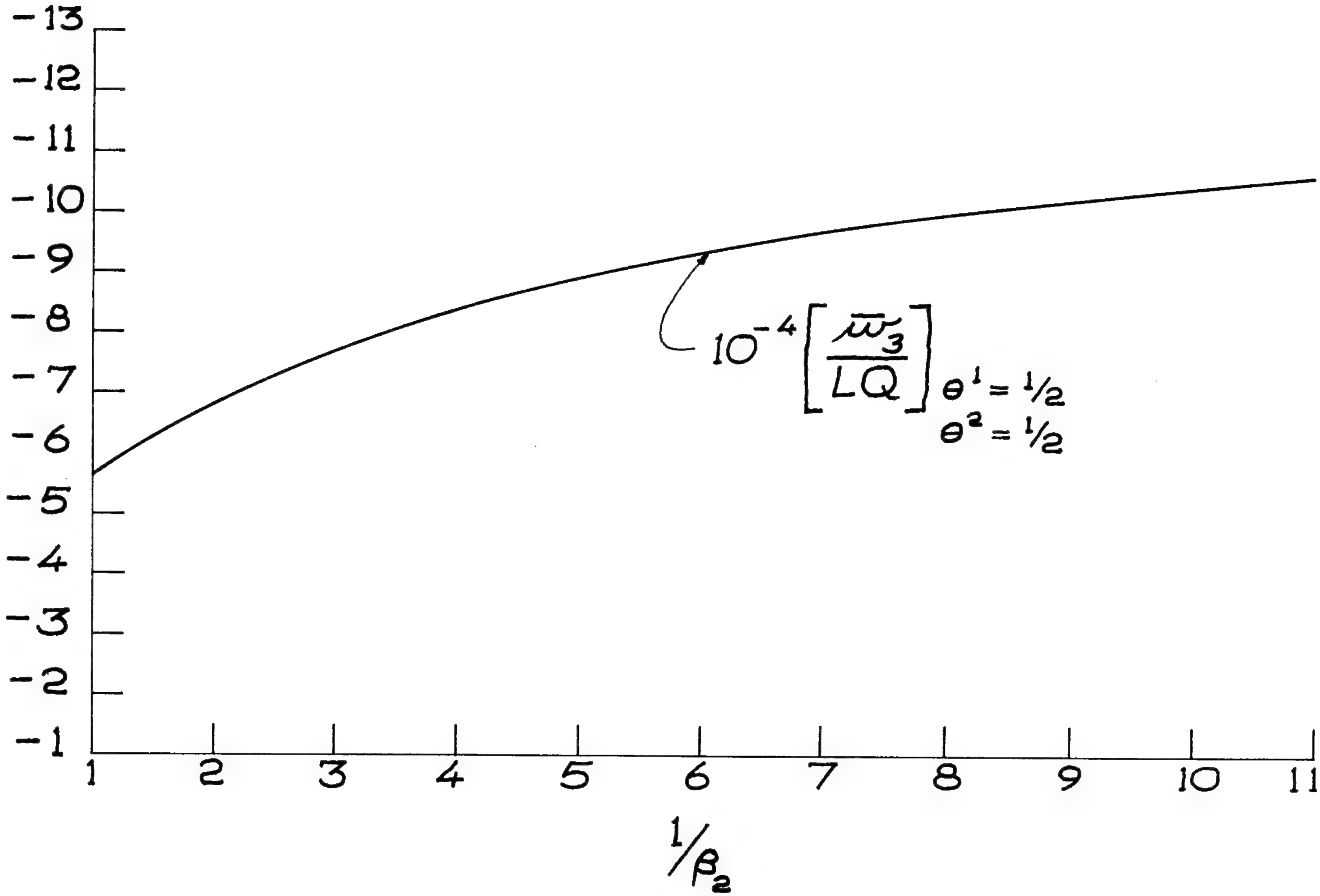

FIG. 4, GROSS DISPLACEMET

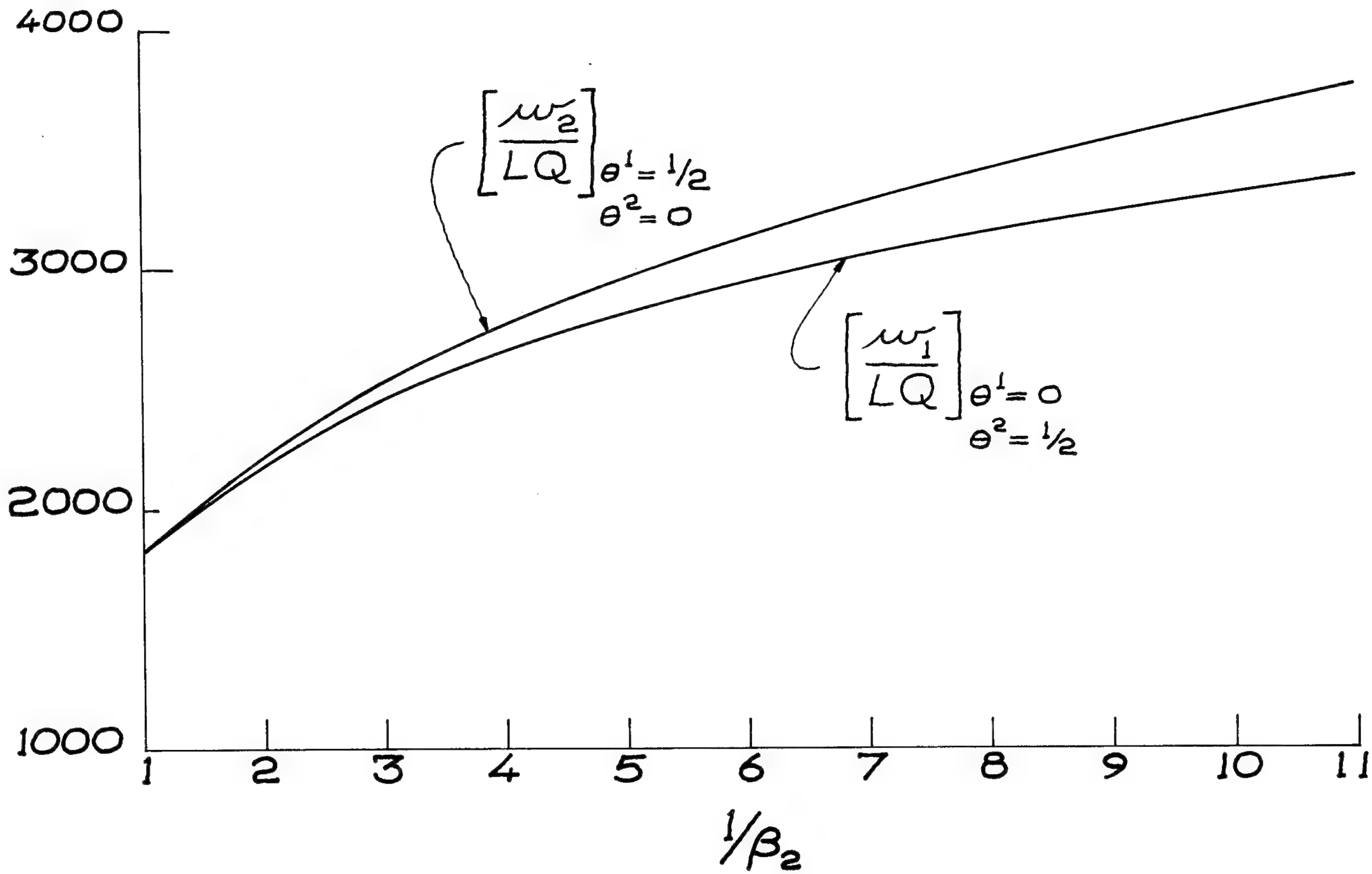

FIG. 5, EDGE ROTATIONS

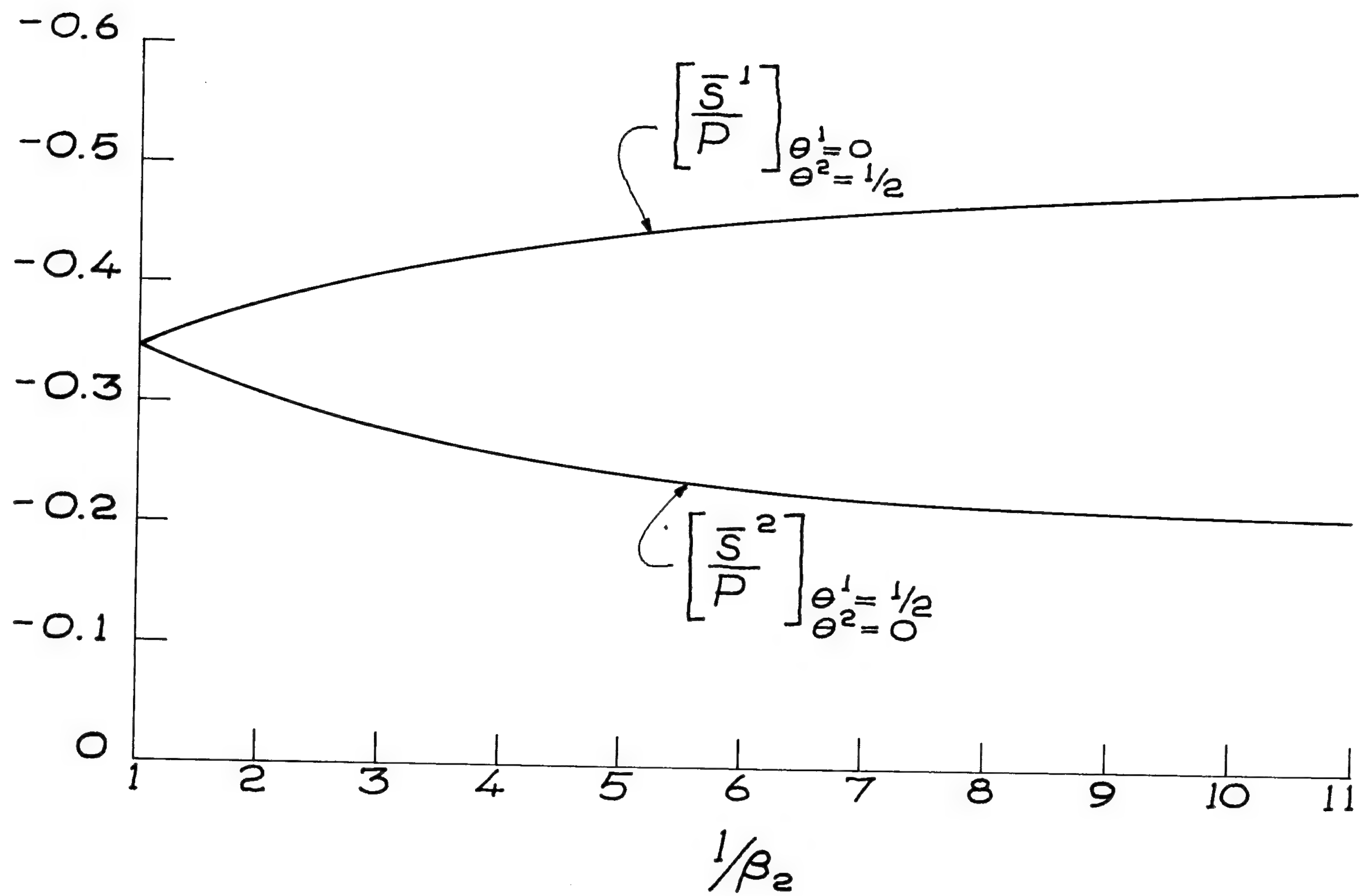

FIG. 6, EDGE SHEARS

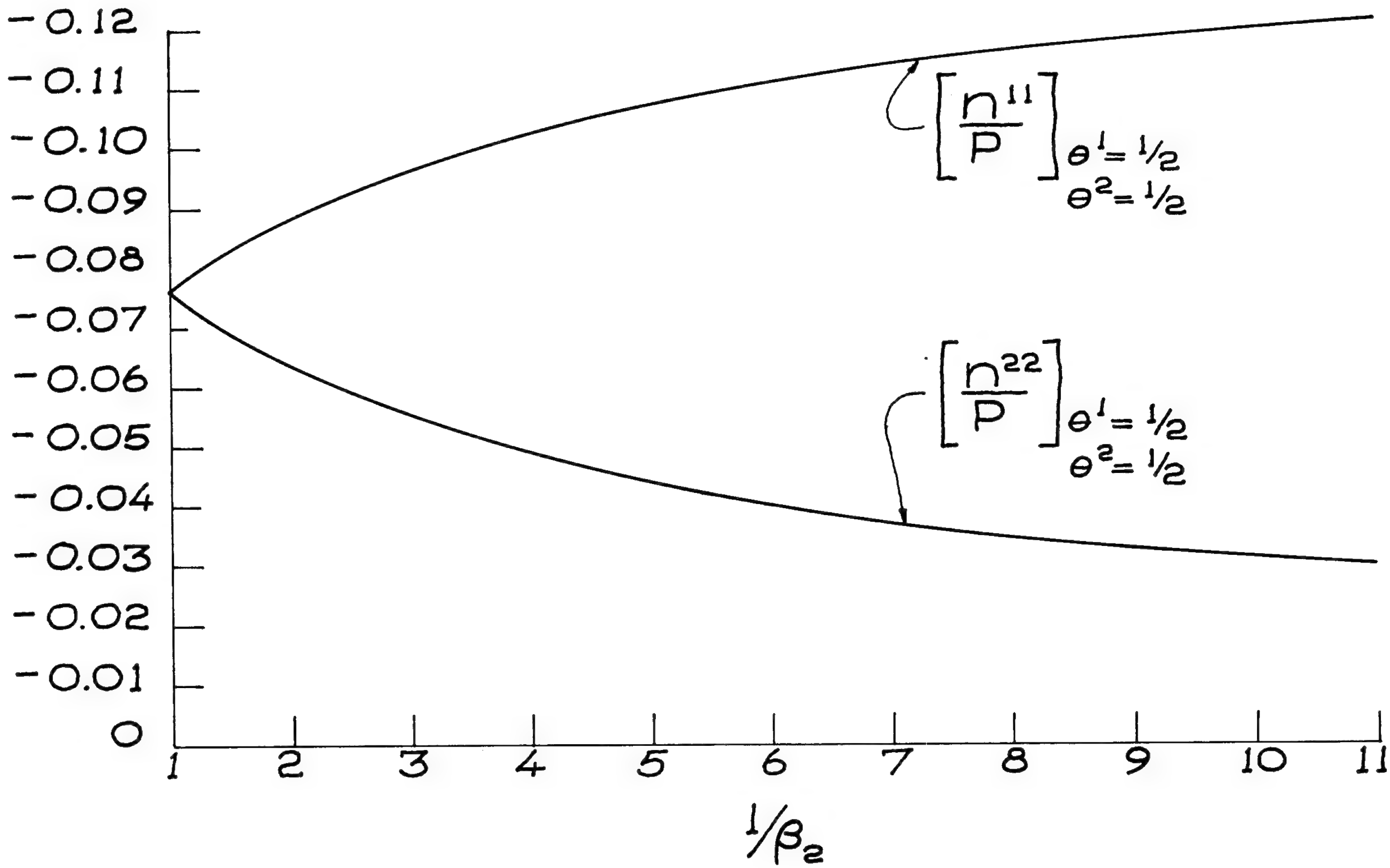

FIG. 7, BENDING MOMENTS

16. Infinite Circular Cylinder with a Viscoelastic Core

In order to study the effect of a viscoelastic core, an infinite circular cylinder loaded by a concentrated uniform ring load acting at $\theta^2 = 0$ is investigated. The facings are isotropic membranes with the same thickness and physical properties. The core is isotropic with an infinite Young's modulus in transverse extension.

The dimensionless surface coordinates are

$$\theta^1 = \phi, \quad \theta^2 = \frac{z}{R}$$

see figure 8. The assumed viscoelastic character of the core is that of a standard linear solid as shown in figure 9. For this example we take

$$\lambda = \frac{1}{80}, \quad {}_2\lambda = \frac{1}{1600},$$

$$G_1/{}_2E = \frac{1}{100}, \quad G_2/{}_2E = \frac{1}{400},$$

$$\nu = \frac{3}{10}.$$

In the sequel the variation of the geometry thru the thickness of the composite shell has been neglected.

From symmetry of the shell and the load

$$\bar{s}^1 = \bar{w}_1 = w_1 = \bar{n}^{12} = n^{12} = 0$$

and all remaining functions are independent of θ^1.

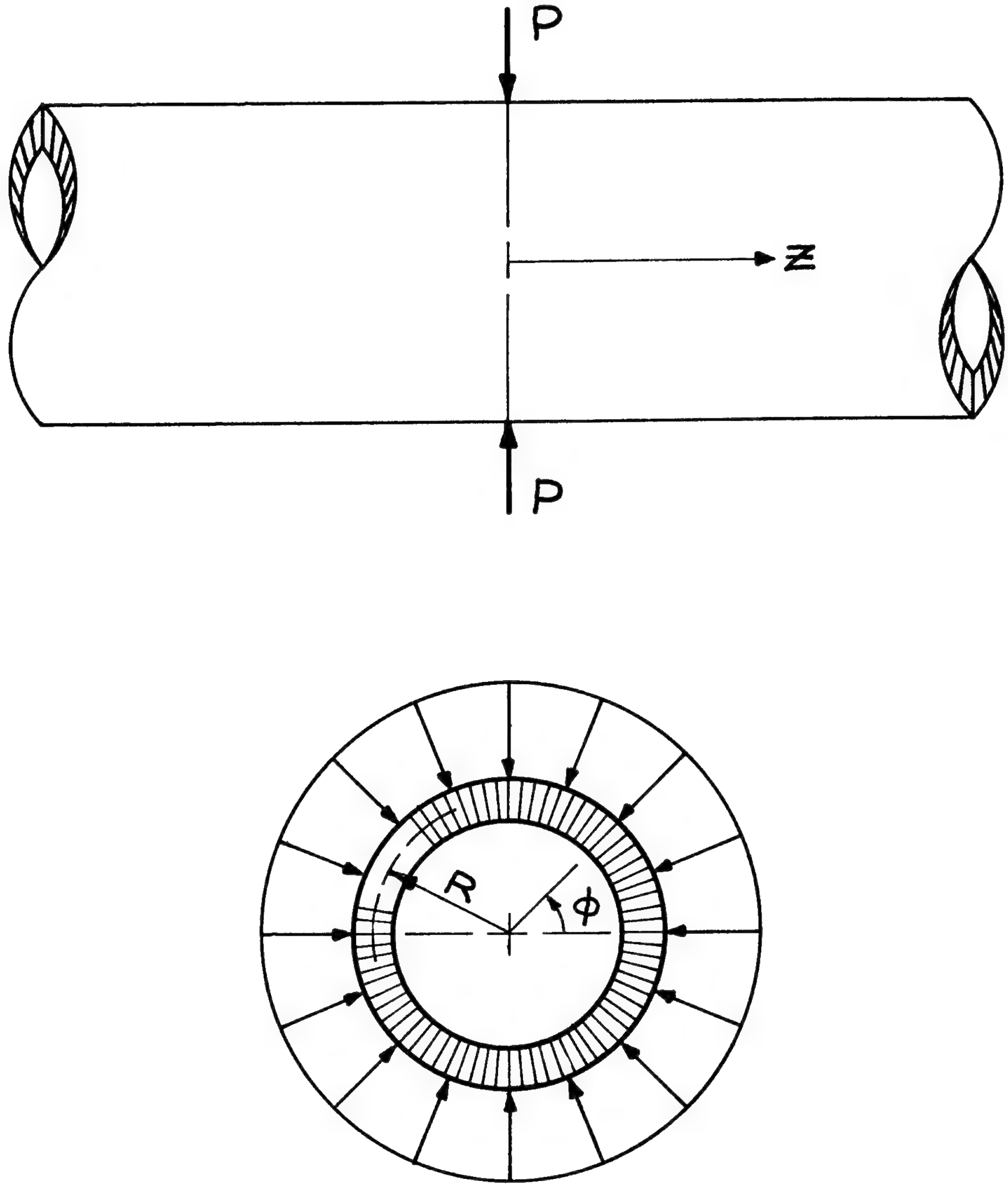

FIG. 8, CIRCULAR CYLINDER

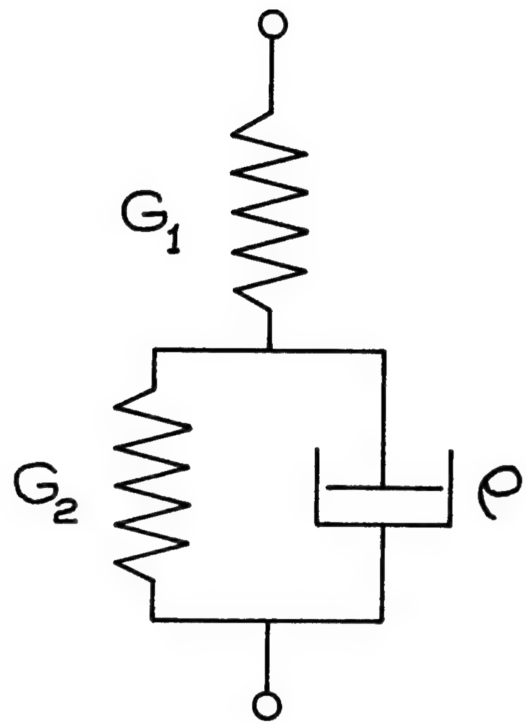

FIG. 9, CORE VISCOELASTIC BEHAVIOR

The following equations are the time Laplace transforms of the equilibrium equations, the core stress-strain relations and the stress resultant-displacement relations for the composite shell

$$-\frac{1}{\eta} P\delta(\theta^2) + {}^*\bar{s}^2_{,2} - {}^*\bar{n}^{11} = 0, \tag{77}$$

$$\frac{L^2}{80}\, {}^*\sigma^{33} + \frac{1}{80\eta} P\delta(\theta^2) - {}^*n^{11} = 0, \tag{78}$$

$${}^*\bar{n}^{22}_{,2} = 0, \tag{79}$$

$${}^*\bar{s}^2 - {}^*n^{22}_{,2} = 0, \tag{80}$$

$${}^*w_2 = -\frac{1}{80}\, {}^*\bar{w}_{3,2} + \frac{{}^*\bar{s}^2}{2L\eta\, {}^*G}, \tag{81}$$

$$\begin{aligned} {}^*\bar{n}^{11} &= \frac{L_2 E}{728}\left[{}^*\bar{w}_3 + \frac{3}{10}\, {}^*\bar{w}_{2,2} \right], \\ {}^*\bar{n}^{22} &= \frac{L_2 E}{728}\left[\frac{3}{10}\, {}^*\bar{w}_3 + {}^*\bar{w}_{2,2} \right], \end{aligned} \tag{82}$$

$$\begin{aligned} {}^*n^{11} &= \frac{3 L_2 E}{1164800}\left[2\, {}^*w_{2,2} - \frac{1}{1600}\, {}^*\bar{w}_{3,22} \right], \\ {}^*n^{22} &= \frac{L_2 E}{116480}\left[2\, {}^*w_{2,2} - \frac{1}{1600}\, {}^*\bar{w}_{3,22} \right]. \end{aligned} \tag{83}$$

Notice that

$${}^*n^{11} = \frac{3}{10}\, {}^*n^{22} \tag{84}$$

The boundary conditions are

$$\left[\overline{w}_2\right]_{\theta^2=0} = 0\,, \tag{85}$$

$$\left[\overline{s}^{2}\right]_{\theta^2=+0} = \frac{P}{2}$$

and $\overline{w}_3$, w_2 and $\overline{w}_{2,2}$ vanish as $\theta^2 \rightarrow \pm\infty$.

From (79), (82) and the boundary conditions one finds

$$\overline{n}^{22} = 0$$

Hence,

$$\overline{w}_{2,2} = -\frac{3}{10}\,\overline{w}_3\,, \tag{86}$$

$$n^{11} = \frac{L_0E}{800}\,\overline{w}_3\,. \tag{87}$$

Differentiating (80) with respect to θ^2 and then using (77), (87) and (81) one obtains

$$\frac{41}{1600}{}^{*}\overline{w}_{3,2222} - \frac{{}_{0}E}{800\eta^{*}G}{}^{*}\overline{w}_{3,22} + \frac{1456}{10}{}^{*}\overline{w}_{3} = -116480\,\frac{P\delta(\theta^2)}{L_0E\eta} + \frac{P}{L\eta^{2*}G}\left[\delta(\theta^2)\right]_{,22}\,, \tag{88}$$

where $\delta(\theta^2)$ is Dirac's delta function.

Taking the θ^2 Fourier transform of (88) yields [13] [14]

$$ {}^{*}\check{w}_3 = -\frac{p\left[\dfrac{116480}{\eta_o E} + \dfrac{\xi^2}{\eta^{2*}G}\right]}{L\sqrt{2\pi}\left[\dfrac{41}{1600}\xi^4 + \dfrac{{}_oE}{800\eta^{*}G}\xi^2 + \dfrac{1456}{10}\right]}, \quad (89) $$

where ξ is the Fourier transform parameter and a $\vee$ over a function indicates a Fourier transform.

The inverse Laplace transform of (89) is

$$ \frac{\check{w}_3}{LQ} = -\frac{128000}{\sqrt{2\pi}}\left\{\frac{\frac{25}{164}\xi^2 + \frac{1456}{41}}{\xi^4 + \frac{1000}{41}\xi^2 + \frac{232960}{41}} + \right. $$

$$ + \left[\frac{\frac{5}{164}\xi^2 + \frac{1456}{41}}{\xi^4 + \frac{200}{41}\xi^2 + \frac{232960}{41}} - \frac{\frac{25}{164}\xi^4 + \frac{1456}{41}}{\xi^4 + \frac{1000}{41}\xi^2 + \frac{232960}{41}}\right] \cdot $$

$$ \left. \cdot \exp\left[\frac{-t}{400\tau}\left(\frac{\xi^4 + \frac{1000}{41}\xi^2 + \frac{232960}{41}}{\xi^4 + \frac{200}{41}\xi^2 + \frac{232960}{41}}\right)\right]\right\} \quad (90) $$

where

$$ Q = \frac{P}{L^2\,{}_oE}, \quad \tau = \frac{\rho}{{}_oE} $$

Observing that $\check{w}_3$ is an even function of ξ it is seen that

$$ \frac{\bar{w}_3}{LQ} = \frac{2}{\sqrt{2\pi}}\int_0^\infty \frac{\check{w}_3}{LQ}(\xi)\cos(\xi\theta^2)\,d\xi \quad (91) $$

Expanding the exponential function

$$ \exp\left[\frac{-t}{400\tau}\left(\frac{\xi^4 + \frac{1000}{41}\xi^2 + \frac{232960}{41}}{\xi^4 + \frac{200}{41}\xi^2 + \frac{232960}{41}}\right)\right] \quad (92) $$

in a power series and comparing the integrals which result from substituting (90) and the power series of (92) into (91), it is seen that in approximating (92) by exp $\left(-t/400\tau\right)$ only a term of order 10^{-3} as compared to one is being neglected when $t/\tau \leq 1200$. With this approximation (91) reduces to

$$\frac{\overline{w}_3}{LQ} = -3020\left(1 - e^{-\frac{t}{400\tau}}\right) e^{-6.62\theta^2} \cdot$$

$$\cdot \left[\cos\left(5.62\theta^2\right) + 0.602 \sin\left(5.62\theta^2\right)\right] -$$

$$-2570\, e^{-\frac{t}{400\tau}}\, e^{-6.24\theta^2}\left[\cos\left(6.04\theta^2\right) +\right.$$

$$\left. + 0.907 \sin\left(6.04\theta^2\right)\right]. \qquad (93)$$

Making the same approximation in the integral form of $\overline{w}_2$ and satisfying (85) one finds

$$\frac{\overline{w}_2}{LQ} = 120 - 120\left(1 - e^{-\frac{t}{400\tau}}\right) e^{-6.62\theta^2} \cdot$$

$$\cdot \left[\cos\left(5.62\theta^2\right) - 0.164 \sin\left(5.62\theta^2\right)\right] -$$

$$-120\, e^{-\frac{t}{400\tau}}\, e^{-6.24\theta^2}\left[\cos\left(6.04\theta^2\right) -\right.$$

$$\left. - 0.0324 \sin\left(6.04\theta^2\right)\right]. \qquad (94)$$

The stress resultant $\overline{n}^{11}$ can be determined from (87) and (93). To determine $\overline{s}^2$ one uses (77), (87) and (86) to obtain

$$\frac{\overline{s}^2{}_{,2}}{p} = \delta(\theta^2) - \frac{1}{240}\left(\frac{\overline{w}_{2,2}}{LQ}\right). \qquad (95)$$

Integrating (95) and using the boundary conditions on $\overline{w}_2$ and $\overline{s}^2$ one finds

$$\frac{\overline{s}^2}{p} = H(\theta^2) - \frac{1}{2} - \frac{1}{240}\left(\frac{\overline{w_2}}{LQ}\right). \qquad (96)$$

Equation (81) can be written

$$\frac{w_2}{LQ} = -\frac{1}{80}\left(\frac{\overline{w}_{3,2}}{LQ}\right) + 50\left(\frac{\overline{s}^2}{p}\right) +$$
$$+ \frac{1}{2\tau}\int_0^t e^{-\frac{1}{400\tau}(t-t')}\left[\frac{\overline{s}^2}{p}(t')\right]dt'. \qquad (97)$$

Again approximating (92) by $\exp\left(-t/400\tau\right)$ (97) becomes

$$\frac{w_2}{LQ} = \left[4.66\left(1-e^{-\frac{t}{400\tau}}\right) - 100\left(\frac{t}{400\tau}\right)e^{-\frac{t}{400\tau}}\right]\cdot$$
$$\cdot e^{-6.62\theta^2}\cos(5.62\theta^2) - \left[382\left(1-e^{-\frac{t}{400\tau}}\right) -\right.$$
$$\left. - 16.4\left(\frac{t}{400\tau}\right)e^{-\frac{t}{400\tau}}\right]e^{-6.62\theta^2}\sin(5.62\theta^2) +$$
$$+\left[0.623 + 100\left(\frac{t}{400\tau}\right)\right]e^{-\frac{t}{400\tau}}e^{-6.24\theta^2}\cos(6.04\theta^2) -$$
$$-\left[377 + 3.24\left(\frac{t}{400\tau}\right)\right]e^{-\frac{t}{400\tau}}e^{-6.24\theta^2}\sin(6.04\theta^2). \qquad (98)$$

Equations (83), (81), (96) and (86) yield

$$\frac{n^{22}}{p} = \frac{1}{116480}\left\{500\delta(\theta^2) - 400e^{-\frac{t}{400\tau}}\delta(\theta^2) -\right.$$
$$- \frac{41}{1600}\left(\frac{\overline{w}_{3,22}}{LQ}\right) + \frac{1}{8}\left(\frac{\overline{w_3}}{LQ}\right) +$$
$$\left. + \frac{1}{800\tau}\int_0^t e^{-\frac{1}{400\tau}(t-t')}\left[\frac{\overline{w_3}}{LQ}(t')\right]dt'\right\}.$$

In evaluating $\varpi_{3,22}$ it is seen that it contains $\delta(\theta^2)$ and that (92) can no longer be approximated by $\exp\left(-t/400\tau\right)$. Here one must use

$$\exp\left[\frac{-t}{400\tau}\left(\frac{\xi^4+\frac{1000}{41}\xi^2+\frac{232960}{41}}{\xi^4+\frac{200}{41}\xi^2+\frac{232960}{41}}\right)\right] \simeq$$

$$\simeq e^{-\frac{t}{400\tau}}\left[1-\left(\frac{t}{400\tau}\right)\left(\frac{\frac{800}{41}\xi^2}{\xi^4+\frac{200}{41}\xi^2+\frac{232960}{41}}\right)\right].$$

After some manipulation one finds

$$100\left(\frac{n^{22}}{p}\right) = -\left[3.78\left(1-e^{-\frac{t}{400\tau}}\right) - \right.$$
$$\left. -1.29\left(\frac{t}{400\tau}\right)e^{-\frac{t}{400\tau}}\right]e^{-6.62\theta^2}\cos(5.62\theta^2) +$$
$$+\left[4.45\left(1-e^{-\frac{t}{400\tau}}\right)+0.778\left(\frac{t}{400\tau}\right)e^{-\frac{t}{400\tau}}\right]\cdot$$
$$\cdot e^{-6.62\theta^2}\sin(5.62\theta^2) - \left[4.28+1.10\left(\frac{t}{400\tau}\right)\right]\cdot$$
$$\cdot e^{-\frac{t}{400\tau}}e^{-6.24\theta^2}\cos(6.04\theta^2) + \left[4.42 - \right.$$
$$\left. -1.00\left(\frac{t}{400\tau}\right)\right]e^{-\frac{t}{400\tau}}e^{-6.24\theta^2}\sin(6.04\theta^2). \quad (99)$$

From (78) and (84) we have

$$\frac{L^2\sigma^{33}}{p} = -\delta(\theta^2) + 24\left(\frac{n^{22}}{p}\right). \qquad (100)$$

Equations (93), (94), (98) and (99) are only valid for $t/\tau \leqslant 1200$ and $\theta^2 \geqslant 0$. For $\theta^2 \leqslant 0$ it is observed that $\bar{w}_3$ and n^{22} are even functions of θ^2 and that $\bar{w}_2$ and w_2 are odd functions of θ^2. Equations (96), (87) and (100) are valid for all t and θ^2.

For $t = \infty$

$$\frac{\bar{w}_3}{LQ} = -3020\, e^{-6.62\theta^2}\left[\cos(5.62\theta^2) + \right.$$
$$\left. + 0.602 \sin(5.62\theta^2)\right],$$
$$\frac{\bar{w}_2}{LQ} = 120 - 120\, e^{-6.62\theta^2}\left[\cos(5.62\theta^2) - \right.$$
$$\left. - 0.164 \sin(5.62\theta^2)\right], \qquad (101)$$
$$\frac{w_2}{LQ} = 4.66\, e^{-6.62\theta^2}\left[\cos(5.62\theta^2) - \right.$$
$$\left. - 82.0 \sin(5.62\theta^2)\right], \qquad (102)$$
$$100\left(\frac{n^{22}}{P}\right) = -3.78\, e^{-6.62\theta^2}\left[\cos(5.62\theta^2) - \right.$$
$$\left. - 1.18 \sin(5.62\theta^2)\right].$$

Equations (98) and (102) show that w_2 is discontinuous at $\theta^2 = 0$ which does not agree with physical reality. From (83) it is seen that w_2 must be discontinuous if n^{22} is to finite at $\theta^2 = 0$. If the facings had been thin shells instead of membranes this inconsistency would not have arisen.

As can be seen from (94) and (101), $\bar{w}_2$ at $\theta^2 = \infty$ is independent of t.

In figures 10 to 14 only the functions for t equal zero and infinity are plotted. The functions at $t/\tau = 1200$ are so close to the functions at $t = \infty$ that they are almost indistinguishable on the figures.

For the numerical values of the physical constants chosen the displacements and stress resultants do not vary greatly as functions of time. However, for a different set of numerical values for the physical constants this may not be true.

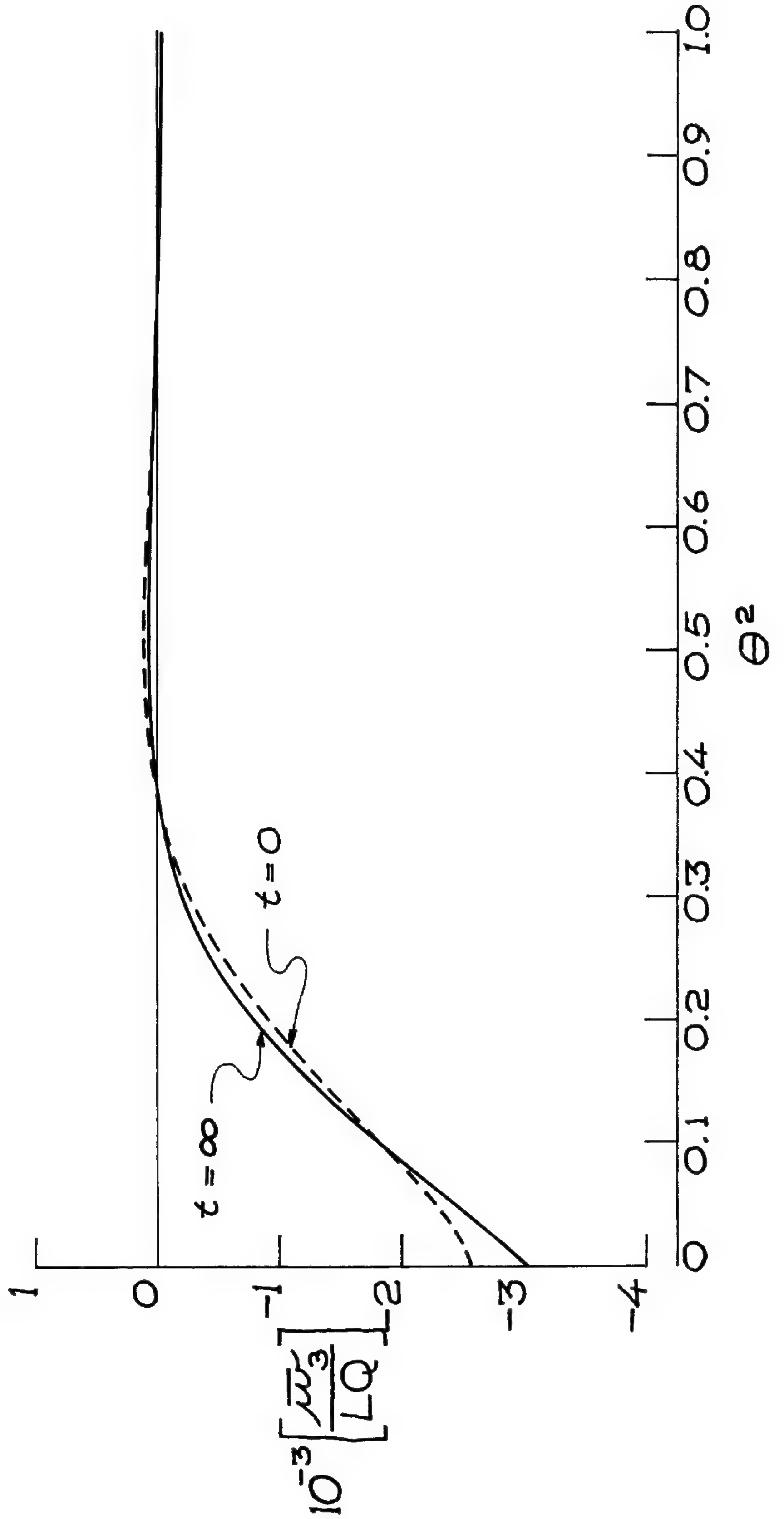

FIG.10, GROSS DISPLACEMET

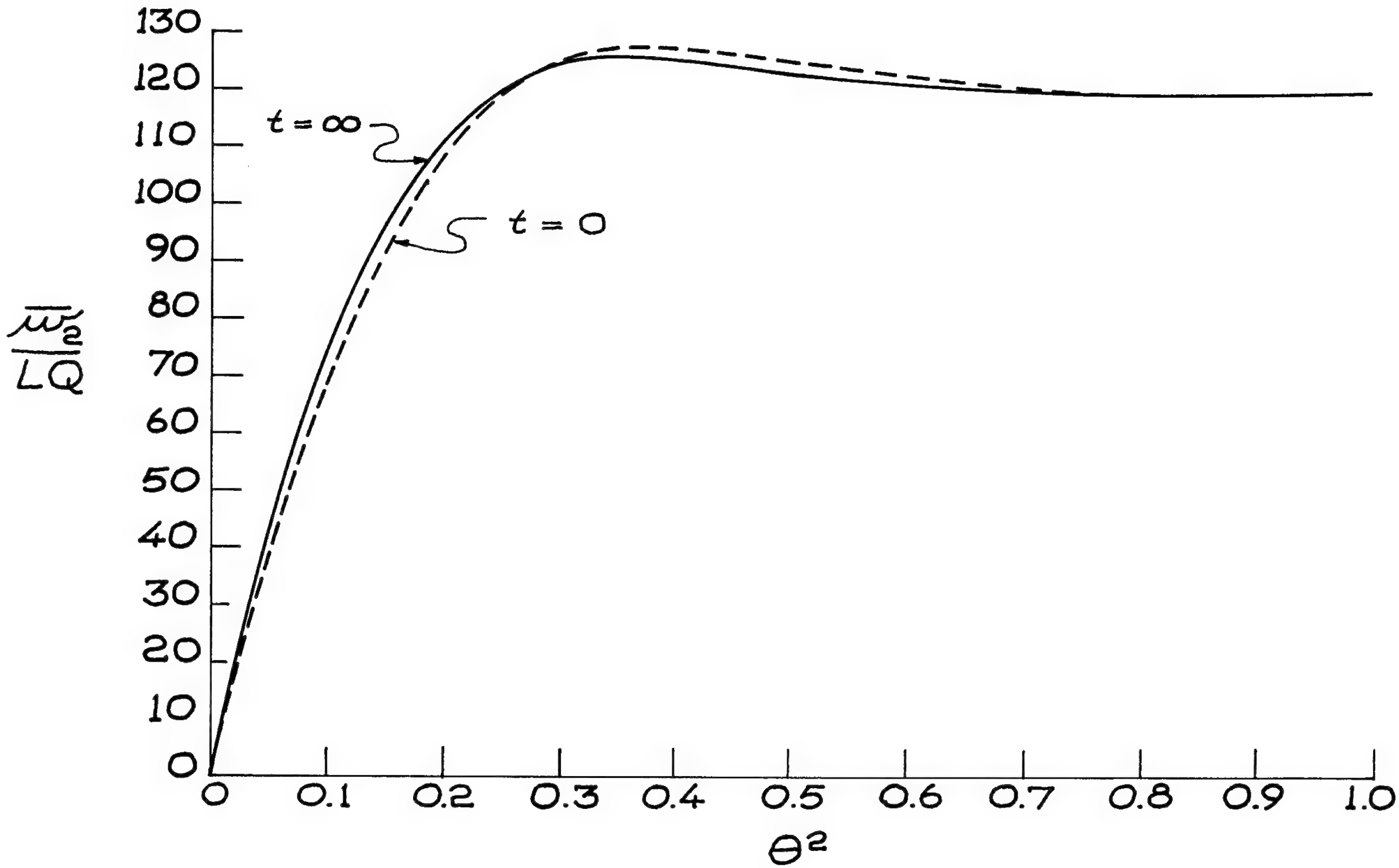

FIG. 11, GROSS AXIAL DISPLACEMET

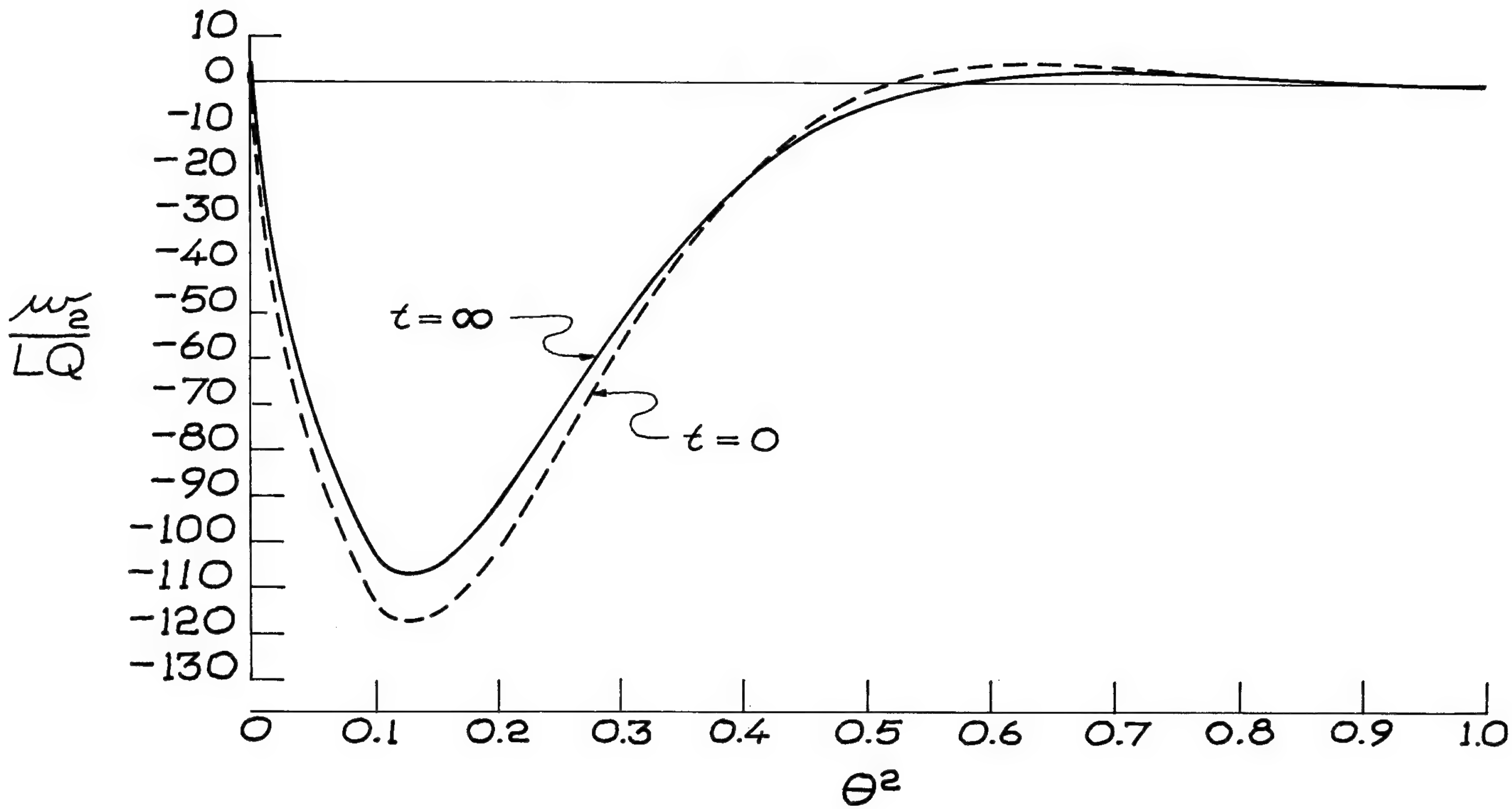

FIG. 12, ROTATION

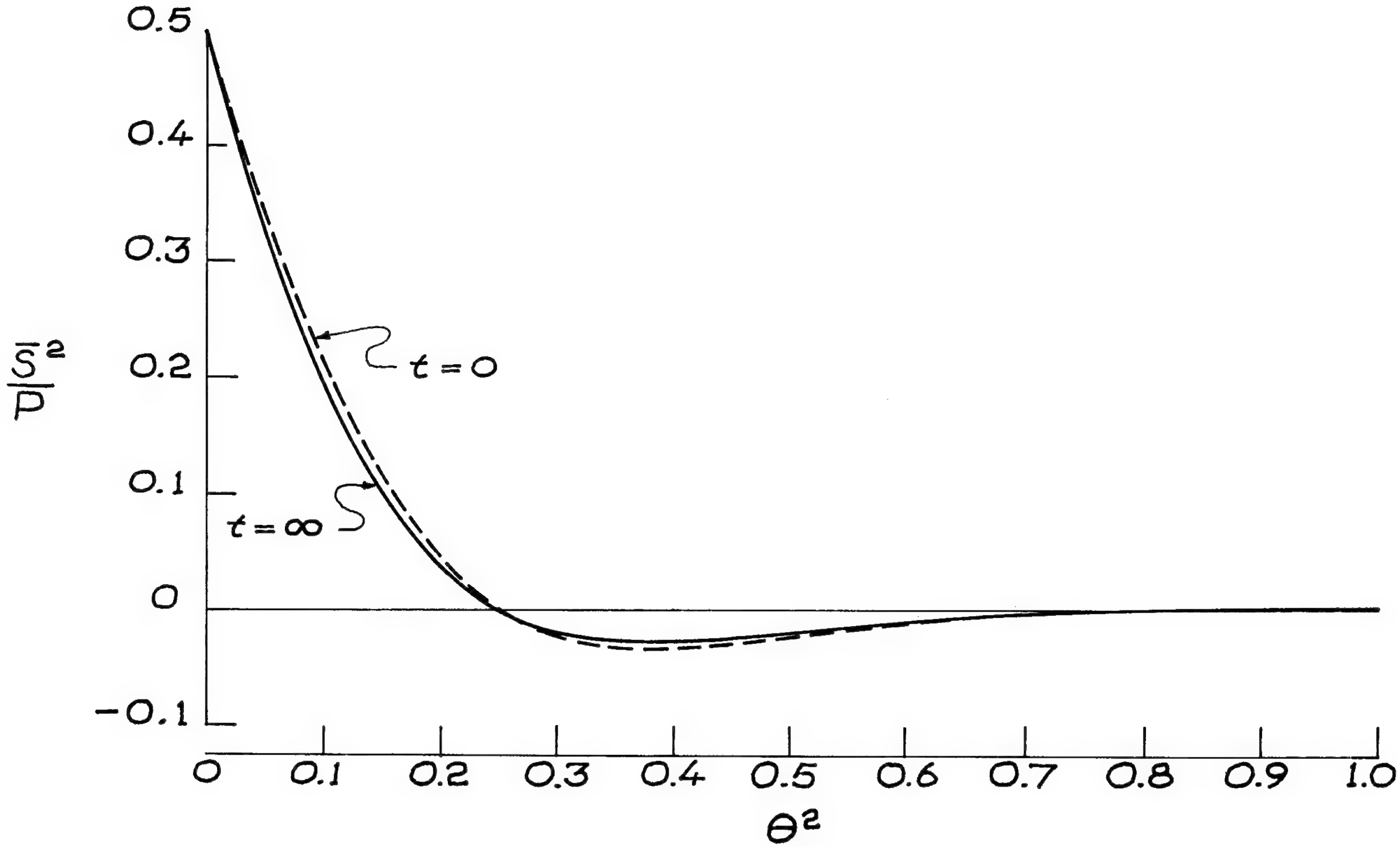

FIG. 13, SHEAR RESULTANT

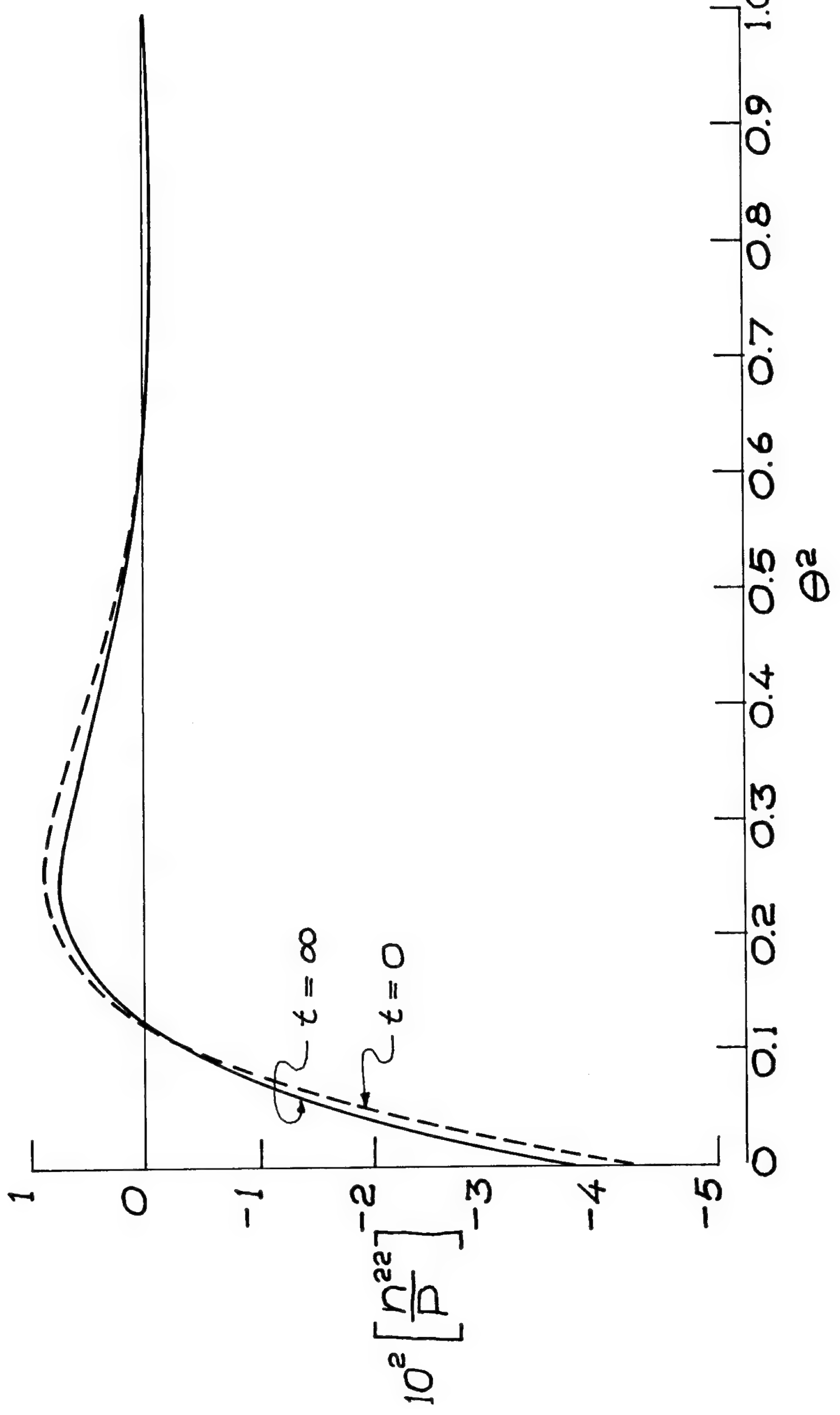

FIG. 14, BENDING MOMENT

NOTATIONS

The tensor notations of [2] are utilized. Latin suffixes take on numbers 1, 2 and 3 while Greek suffixes take on numbers 1 and 2. Repeated indices are not summed when enclosed by parentheses. The prefix $\underline{n}$ stands for $\underline{0}$ or $\underline{1}$ according as the quantity is associated with the upper or lower facing, respectively. If two signs appear, i.e. $\pm\, n^{\alpha\beta}$, the upper (or lower) sign applies whenever reference is being made to the upper (or lower) facing. A comma denotes partial differentiation, i.e. $h_{,\alpha} = \frac{\partial h}{\partial \theta^{\alpha}}$. A vertical bar ($|$) denotes covariant differentiation with respect to the three dimensional space while a double vertical bar ($\|$) denotes covariant differentiation with respect to the core mid-surface coordinates.

Symbol	Description
L	a characteristic length of the core mid-surface
d	thickness of the core
$_{\underline{n}}d$	thickness of a facing, $\underline{n} = \underline{0}$ or $\underline{1}$
λ	$d/2L$
$_{\underline{n}}\lambda$	$_{\underline{n}}d/L$
θ^{α}	dimensionless surface coordinates, lines of curvature
$\theta^{3}, {}_{\underline{n}}\theta^{3}$	dimensionless normal coordinates
$\vec{a}_{\alpha}$	base vectors $\vec{\rho}_{,\alpha}$ where $\vec{\rho}$ is the dimensionless position vector of the core mid-surface
$\hat{a}_{3}$	unit normal to core mid-surface

${}_{\underline{n}}\vec{a}_\alpha$	dimensionless interface base vectors (see $\vec{a}_\alpha$)
${}_{\underline{n}}\hat{a}_3$	$\hat{a}_3$
b_α^β	coefficients of the second fundamental form for the core mid-surface
${}_{\underline{n}}b_\alpha^\beta$	coefficients of the second fundamental form for the interface surfaces
$\vec{g}_\alpha$	$L(\delta_\alpha^\beta - \lambda\theta^3 b_\alpha^\beta)\vec{a}_\beta$, base vectors
$\vec{g}_3$	$\lambda L \hat{a}_3$
${}_{\underline{n}}\vec{g}_3$	$({}_{\underline{n}}\lambda/\lambda)\,\vec{g}_3$
$a_{\alpha\beta}$	$\vec{a}_\alpha \cdot \vec{a}_\beta$
$g_{\alpha\beta}$	$\vec{g}_\alpha \cdot \vec{g}_\beta$
a	determinant $\lvert a_{\alpha\beta}\rvert$
g	determinant $\lvert g_{\alpha\beta}\rvert$
${}_{\underline{n}}\epsilon_{\alpha\beta}$	$\sqrt{{}_{\underline{n}}a}\, e_{\alpha\beta 3}$ where e_{rst} is the permutation symbol
${}_{\underline{n}}\hat{A}_3$	deformed unit normal to interface surfaces
$\hat{n}$	$u_\alpha \vec{a}^\alpha$ unit normal to core edge at the mid-surface
${}_{\underline{n}}\hat{n}$	${}_{\underline{n}}u_{\alpha\,\underline{n}}\vec{a}^\alpha$ unit normal to the edge of a facing at the interface
${}_{\underline{n}}\hat{t}$	${}_{\underline{n}}t_{\alpha\,\underline{n}}\vec{a}^\alpha$ unit tangent to the edge of a facing at the interface
h	mean curvature of core mid-surface
k	Gaussian curvature of core mid-surface
$\bar{\alpha}$	$1/L^3_{\underline{2}}\, j_{\underline{2}} d + 1/L^3_{\underline{1}}\, j_{\underline{1}} d$
α	$1/L^3_{\underline{2}}\, j_{\underline{2}} d - 1/L^3_{\underline{1}}\, j_{\underline{1}} d$

$\bar{\beta}$ ———— $1/L_{\underline{0}}^2 j_{\underline{0}} d^2 + 1/L_{\underline{1}}^2 j_{\underline{1}} d^2$

β ———— $1/L_{\underline{0}}^2 j_{\underline{0}} d^2 - 1/L_{\underline{1}}^2 j_{\underline{1}} d^2$

$\bar{\gamma}$ ———— $1/L_{\underline{0}} j_{\underline{0}} d^3 + 1/L_{\underline{1}} j_{\underline{1}} d^3$

γ ———— $1/L_{\underline{0}} j_{\underline{0}} d^3 - 1/L_{\underline{1}} j_{\underline{1}} d^3$

${}_{\underline{n}}\Gamma^{\gamma}_{\alpha\beta}$ ———— Christoffel symbols of the second kind evaluated at the interface surfaces

$\lambda\, {}_{\underline{n}}\Upsilon^{\gamma}_{\alpha\beta}$ ———— ${}_{\underline{n}}\Gamma^{\gamma}_{\alpha\beta} - [\Gamma^{\gamma}_{\alpha\beta}]_{\theta^3=0}$

$2\bar{\Upsilon}^{\gamma}_{\alpha\beta}$ ———— ${}_{\underline{0}}\Upsilon^{\gamma}_{\alpha\beta} + {}_{\underline{1}}\Upsilon^{\gamma}_{\alpha\beta}$

$2\lambda\Upsilon^{\gamma}_{\alpha\beta}$ ———— ${}_{\underline{0}}\Upsilon^{\gamma}_{\alpha\beta} - {}_{\underline{1}}\Upsilon^{\gamma}_{\alpha\beta}$

τ ———— volume of a body

σ ———— surface of a body

S ———— core mid-surface

${}_{\underline{n}}S$ ———— interface surfaces

${}_{\underline{n}}\bar{S}$ ———— exterior faces of the facings

σ_1 ———— part of σ on which the stresses are prescribed

σ_2 ———— part of σ on which the displacements are prescribed

${}_{\underline{n}}\Omega_1$ ———— σ_1 for the edges of the facings

${}_{\underline{n}}\Omega_2$ ———— σ_2 for the edges of the facings

${}_{\underline{n}}C_1$ ———— part of interface boundary curves on which the stress resultants are prescribed

${}_{n}C_2$	part of interface boundary curves on which the displacements are prescribed
${}_{n}s$	dimensionless arc length along interface boundary curves
${}_{n}n$	dimensionless arc length along the normals to the edges of the facings at the interfaces
$\vec{V}$	$V_r \vec{g}^{\,r}$, displacement vector
$2e_{rs}$	$V_{r\|s} + V_{s\|r}$
$2\omega_{rs}$	$V_{r\|s} - V_{s\|r}$
${}_{n}\vec{V}$	interface displacement vector
$2\bar{w}_r$	$\vec{a}_r \cdot ({}_{0}\vec{V} + {}_{1}\vec{V})$ average displacement of the interfaces
$2w_r$	$\vec{a}_r \cdot ({}_{0}\vec{V} - {}_{1}\vec{V})$ relative displacement of the interfaces
γ_{rs}	strain tensor
τ^{rs}	stress tensor
$\bar{s}^{\alpha}$	$\dfrac{\lambda L}{\sqrt{a}} \displaystyle\int_{-1}^{+1} \sqrt{g}\, \tau^{3\alpha} d\theta^3$
σ^{33}	$\dfrac{\lambda^2 L^2}{\sqrt{a}} \left[\sqrt{{}_{0}a}\, {}_{0}\tau^{33} + \sqrt{{}_{1}a}\, {}_{1}\tau^{33} \right]$
$\bar{n}^{\alpha\beta}$	${}_{0}n^{\alpha\beta} + {}_{1}n^{\alpha\beta}$
$n^{\alpha\beta}$	$\lambda ({}_{0}n^{\alpha\beta} - {}_{1}n^{\alpha\beta})$
$\bar{m}^{\alpha\beta}$	${}_{0}m^{\alpha\beta} + {}_{1}m^{\alpha\beta}$
$m^{\alpha\beta}$	$\lambda ({}_{0}m^{\alpha\beta} - {}_{1}m^{\alpha\beta})$
$\bar{p}^{r}$	${}_{0}p^{r} + {}_{1}p^{r}$
p^{r}	${}_{0}p^{r} - {}_{1}p^{r}$

E	Young's modulus for transverse extension of the core
E^{α}	shear modulus of an orthotropic core
G	shear modulus of an isotropic core
${}_{n}E$	Young's modulus of an isotropic facing
ν	Poisson's ratio for both facings
${}_{n}B^{\alpha\beta\gamma\eta}$	elastic coefficients for a facing
$2\overline{B}^{\alpha\beta\gamma\eta}$	${}_{0}B^{\alpha\beta\gamma\eta} + {}_{1}B^{\alpha\beta\gamma\eta}$
$2B^{\alpha\beta\gamma\eta}$	${}_{0}B^{\alpha\beta\gamma\eta} - {}_{1}B^{\alpha\beta\gamma\eta}$
${}_{n}C_{\alpha\beta\gamma\eta}$	elastic coefficients defined by ${}_{n}B^{\gamma\eta\alpha\beta}\,{}_{n}C_{\gamma\eta\xi\mu} = \frac{1}{2}\left(\delta^{\alpha}_{\xi}\delta^{\beta}_{\mu} + \delta^{\alpha}_{\mu}\delta^{\beta}_{\xi}\right)$
$2\overline{C}_{\alpha\beta\gamma\eta}$	${}_{0}C_{\alpha\beta\gamma\eta} + {}_{1}C_{\alpha\beta\gamma\eta}$
$2C_{\alpha\beta\gamma\eta}$	${}_{0}C_{\alpha\beta\gamma\eta} - {}_{1}C_{\alpha\beta\gamma\eta}$
W	$\frac{1}{2}\,{}_{n}C_{rskl}\,\tau^{rs}\tau^{kl}$
$\sim$	a symbol placed over a quantity indicating that the quantity is prescribed on the edge of a facing
$*$	a symbol placed over a function of time to indicate the Laplace transform of the function
η	Laplace transform parameter
${}_{n}j$	$\sqrt{\dfrac{{}_{n}a}{a}} = 1 \mp 2\lambda h + \lambda^{2} k$

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